

# Dividing Polynomials: Remainder and Factor Theorems

Grade 11 – Algebra 2 math · Lesson · Standards: A-APR.B.2, A-APR.D.6, A-APR.C.4

**Quick answer:** To divide polynomials, repeatedly divide leading terms, multiply, and subtract; for a divisor  $x - r$ , synthetic division does the same with coefficients. The Remainder Theorem says the remainder equals  $f(r)$ , and the Factor Theorem says  $x - r$  is a factor exactly when  $f(r) = 0$ .

**Standards:** A-APR.B.2, A-APR.D.6, A-APR.C.4 · **Level:** Grade 11 Algebra 2

## How does polynomial long division work?

Dividing polynomials follows the same pattern as dividing whole numbers. For a polynomial  $f(x)$  and a divisor  $d(x)$  there are a **quotient**  $q(x)$  and a **remainder**  $r(x)$  with

$$f(x) = d(x) \cdot q(x) + r(x)$$

where the degree of  $r$  is less than the degree of  $d$ .

### Method: long division

1. Write both polynomials in descending order, adding 0 for any missing term.
2. Divide the leading term of the dividend by the leading term of the divisor.
3. Multiply the divisor by that result and subtract.
4. Repeat with the new line until the remainder has a lower degree than the divisor.

## What is synthetic division?

When the divisor has the form  $x - r$ , synthetic division is a shortcut that uses only the coefficients. Bring down the first coefficient, multiply by  $r$ , add to the next coefficient, and repeat.

|          |           |           |            |           |
|----------|-----------|-----------|------------|-----------|
| <b>3</b> | 1         | −7        | 4          | 12        |
|          |           | 3         | −12        | −24       |
| <b>1</b> | <b>−4</b> | <b>−8</b> | <b>−12</b> |           |
|          |           |           |            | remainder |

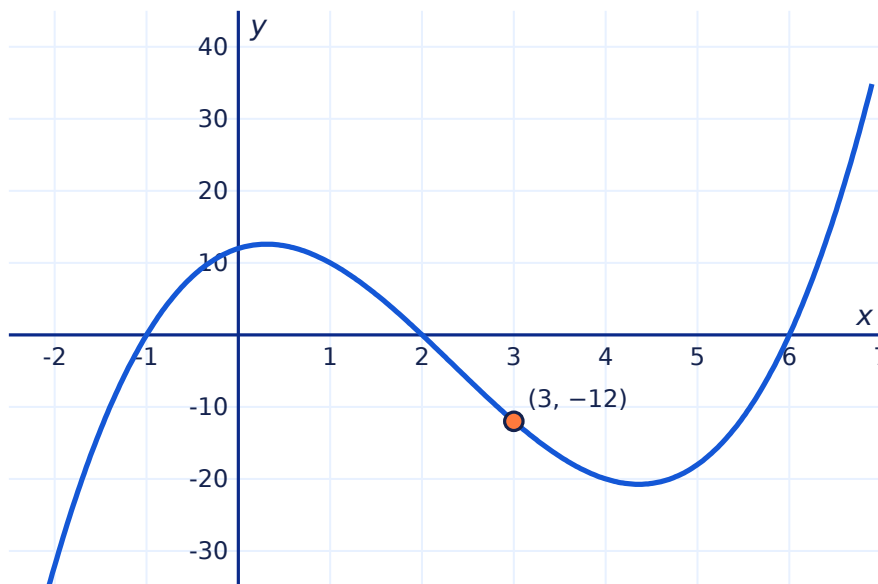
*Dividing  $x^3 - 7x^2 + 4x + 12$  by  $x - 3$  gives the quotient  $x^2 - 4x - 8$  and remainder  $-12$ .*

## What do the Remainder and Factor Theorems say?

### Remainder and Factor Theorems

**Remainder Theorem:** when  $f(x)$  is divided by  $x - r$ , the remainder is  $f(r)$ .

**Factor Theorem:**  $x - r$  is a factor of  $f(x)$  exactly when  $f(r) = 0$ .



*The remainder when dividing by  $x - 3$  is the height of the graph at  $x = 3$ , which is  $-12$ .*

### Worked example

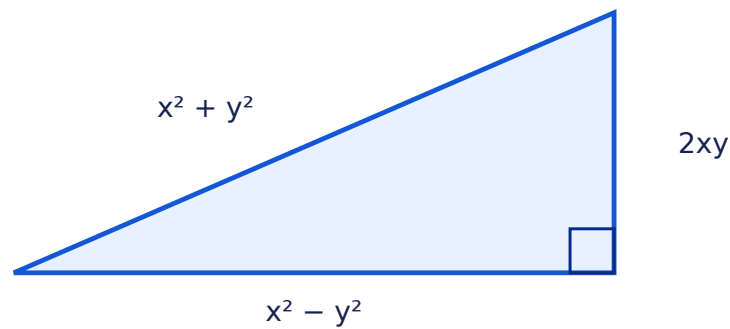
For  $f(x) = x^3 - 7x^2 + 4x + 12$ , compute  $f(3) = 27 - 63 + 12 + 12 = -12$ . The remainder on division by  $x - 3$  is **-12**, so  $x - 3$  is not a factor.

## Which polynomial identities are worth knowing?

An identity is true for every value of the variables. Expanding both sides proves it. A useful one is

$$(x^2 + y^2)^2 = (x^2 - y^2)^2 + (2xy)^2$$

Because of it, any whole numbers  $x > y > 0$  produce a Pythagorean triple.



The three expressions are the sides of a right triangle for any  $x > y > 0$ .

### Common mistake

Forgetting zeros for missing terms. To divide  $x^4 - 5x^2 + 2$  synthetically, use the coefficients 1, 0, -5, 0, 2. Skipping the zeros shifts every later column.

### Mathbot's tip

Check any division by multiplying back: divisor times quotient plus remainder must rebuild the original polynomial.

### Key points to remember

- $f(x) = d(x) \cdot q(x) + r(x)$ , and the degree of  $r$  is less than the degree of  $d$ .
- Synthetic division works only for divisors of the form  $x - r$ .
- Remainder Theorem: dividing  $f(x)$  by  $x - r$  leaves the remainder  $f(r)$ .
- Factor Theorem:  $f(r) = 0$  exactly when  $x - r$  is a factor of  $f(x)$ .
- Polynomial identities such as  $(x^2 + y^2)^2 = (x^2 - y^2)^2 + (2xy)^2$  can be proved by expanding.

## Dividing Polynomials: Remainder and Factor Theorems: frequently asked questions

### When can you use synthetic division?

Synthetic division works when the divisor is a linear expression with leading coefficient 1, written  $x - r$ . For a divisor such as  $2x - 3$ , divide by  $x - \frac{3}{2}$  and then divide the quotient by 2.

### What is the quickest way to find a remainder?

Use the Remainder Theorem: substitute  $r$  into  $f(x)$ . This gives the same number that long division or synthetic division would leave, without finding the quotient.

### How does the Factor Theorem help solve equations?

If  $f(r) = 0$ , then  $x - r$  is a factor. Divide it out to get a lower-degree polynomial, then factor again to

find the other zeros.

### **Why do missing terms need a zero coefficient?**

Each column in synthetic division stands for one power of  $x$ . A missing power still takes a column, so its coefficient 0 must be written.

### **How can you prove a polynomial identity?**

Expand one side or both sides using the distributive property and combine like terms. If the two sides have identical coefficients, the identity is true for every value.