

Rational Functions: Asymptotes, Holes, and Graphs

Grade 11 – Algebra 2 math · Lesson · Standards: F-IF.C.7d, F-IF.B.4

Quick answer: A rational function is a quotient of polynomials. Its graph has a vertical asymptote where a denominator factor does not cancel, a hole where a factor cancels, and a horizontal asymptote set by the degrees. The form $y = a/(x - h) + k$ has asymptotes $x = h$ and $y = k$.

Standards: F-IF.C.7d, F-IF.B.4 · **Level:** Grade 11 Algebra 2

What is a rational function and where is it undefined?

A **rational function** is a quotient of two polynomials, $f(x) = \frac{p(x)}{q(x)}$. Its domain is every real number except the zeros of the denominator q . At each such number the graph has either a vertical asymptote or a hole.

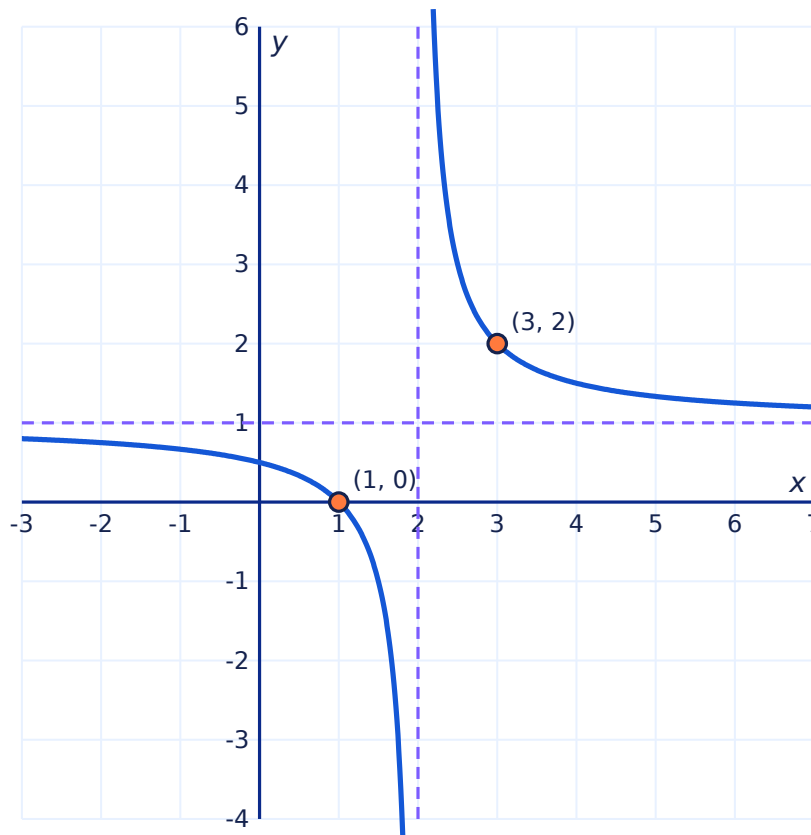
Asymptotes and holes

A **vertical asymptote** is a vertical line the graph approaches but never reaches; it occurs at a zero of the denominator that does not cancel. A **hole** occurs at a zero that cancels with the numerator. A **horizontal asymptote** describes the behavior of $f(x)$ as x becomes very large in either direction.

Degrees of p and q	Horizontal asymptote
Degree of $p <$ degree of q	$y = 0$
Degree of $p =$ degree of q	$y = (\text{leading coefficient of } p) / (\text{leading coefficient of } q)$
Degree of $p >$ degree of q	None

How do you graph $y = a/(x - h) + k$?

This form is the parent function $y = 1/x$ stretched by a , shifted right h and up k . The vertical asymptote is $x = h$ and the horizontal asymptote is $y = k$. If $a > 0$ the two branches lie in the upper right and lower left of the asymptote cross; if $a < 0$ they lie in the other two regions.



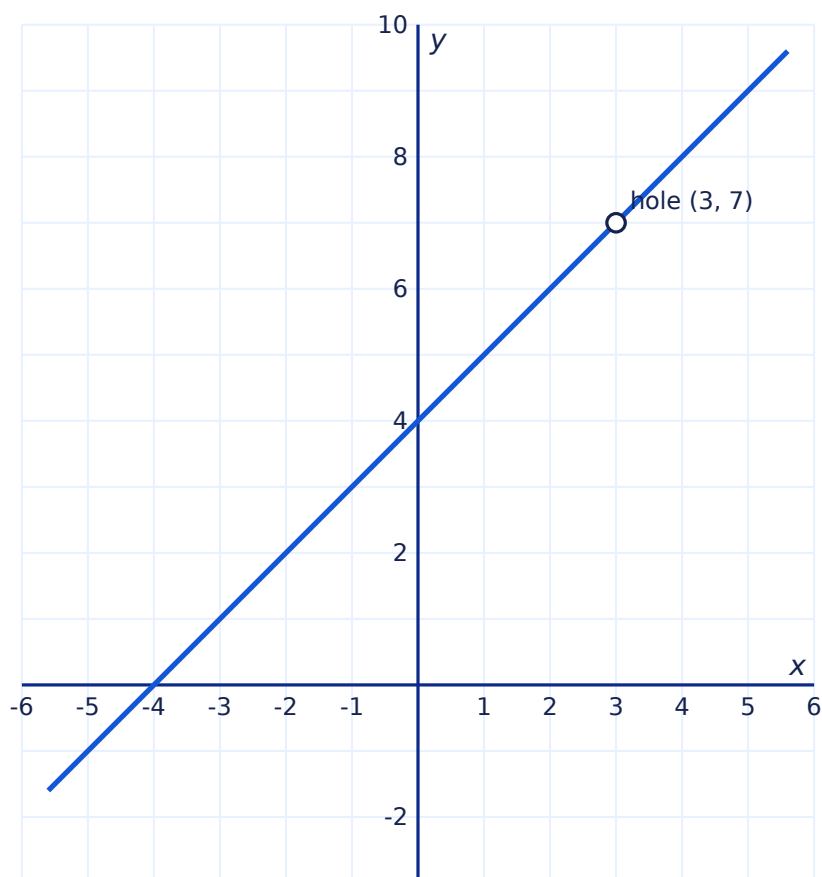
$y = \frac{1}{x-2} + 1$ has asymptotes $x = 2$ and $y = 1$.

Method: graph a rational function

1. Factor the numerator and denominator and cancel common factors; each canceled factor is a hole.
2. Draw the vertical asymptotes at the remaining zeros of the denominator.
3. Find the horizontal asymptote from the degrees.
4. Plot the intercepts and a few points, then sketch each branch.

Worked example

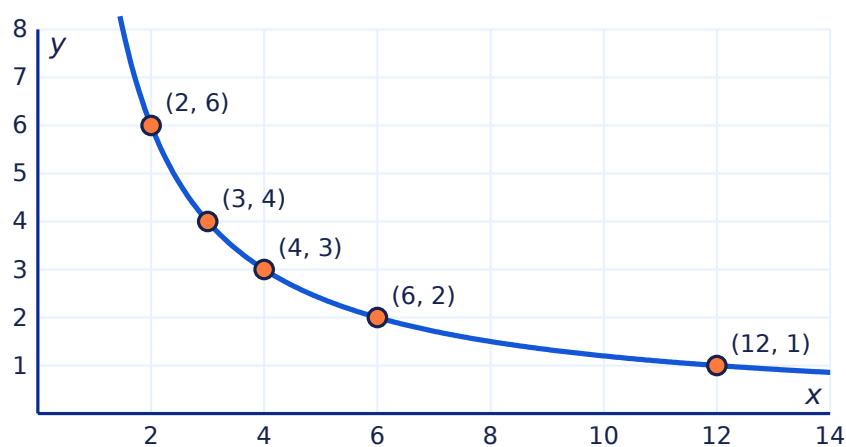
$f(x) = \frac{(x^2 + x - 12)}{(x - 3)} = \frac{((x + 4)(x - 3))}{(x - 3)}$. The factor $x - 3$ cancels, so $f(x) = x + 4$ with a hole at $x = 3$. The hole's height is $3 + 4 = 7$, so the hole is at **(3, 7)**.



A hole is drawn as an open circle on the simplified graph.

What is inverse variation?

Two quantities vary **inversely** when their product stays constant: $xy = k$, or $y = \frac{k}{x}$. When one quantity doubles, the other halves. The graph is a rational function with asymptotes at the axes.



For $y = \frac{12}{x}$, doubling x halves y .

Common mistake

Do not confuse a hole with an asymptote: if a factor cancels, there is a hole, not an asymptote. Also, a graph can cross a horizontal asymptote; only vertical asymptotes are never crossed.

Mathbot's tip

Cancel first, ask questions later. Everything that cancels is a hole; everything left in the denominator is an asymptote.

Key points to remember

- The domain excludes every zero of the denominator.
- A canceled factor creates a hole; a remaining denominator factor creates a vertical asymptote.
- The horizontal asymptote depends on the degrees: smaller top gives $y = 0$, equal degrees give the ratio of leading coefficients.
- $y = a/(x - h) + k$ has asymptotes $x = h$ and $y = k$.
- Inverse variation means $xy = k$, so the graph is $y = k/x$.

Rational Functions: Asymptotes, Holes, and Graphs: frequently asked questions

What is the difference between a hole and a vertical asymptote?

Both come from a zero of the denominator. If the factor also cancels with the numerator, the graph has a hole at that x . If the factor does not cancel, the graph has a vertical asymptote there.

How do you find a horizontal asymptote?

Compare the degrees of the numerator and denominator. If the numerator has the smaller degree, the asymptote is $y = 0$. If the degrees are equal, it is the ratio of the leading coefficients. If the numerator has the larger degree, there is none.

What is the domain and range of $y = a/(x - h) + k$?

The domain is all real numbers except $x = h$. The range is all real numbers except $y = k$. Both excluded values are the equations of the asymptotes.

How do you recognize inverse variation?

The product of the two quantities is always the same number. A table in which x and y change in opposite directions with a constant product describes inverse variation.

Can a graph cross its horizontal asymptote?

Yes, for some rational functions the graph crosses a horizontal asymptote at a finite point. The asymptote describes the behavior for very large positive or negative x , so a crossing in the middle is allowed.