

# Using Derivatives to Analyze Graphs and Optimize

Grade 12 – AP Calculus AB math · Lesson · Standards: AP Calculus AB Unit 5

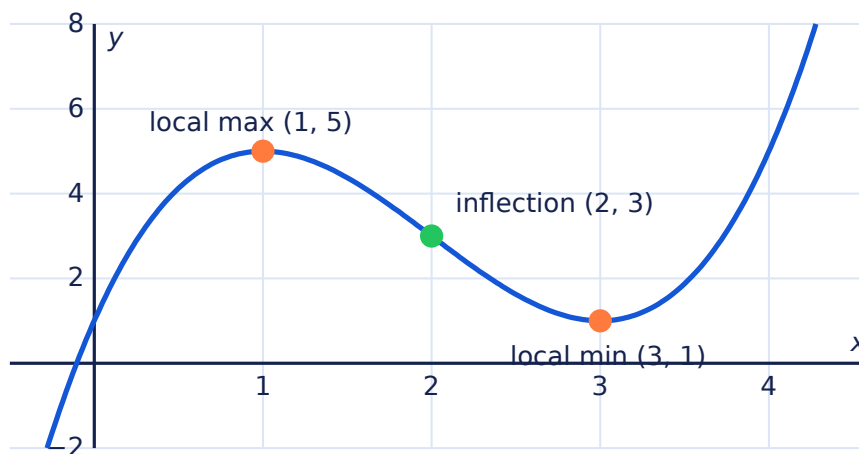
**Quick answer:** Derivatives reveal the shape of a graph: the sign of the first derivative shows where a function rises or falls, the sign of the second derivative shows concavity, and critical points locate extrema. The Mean Value and Extreme Value Theorems guarantee when such points exist on an interval.

**Standards:** AP Calculus AB Unit 5 · **Level:** Grade 12 AP Calculus AB

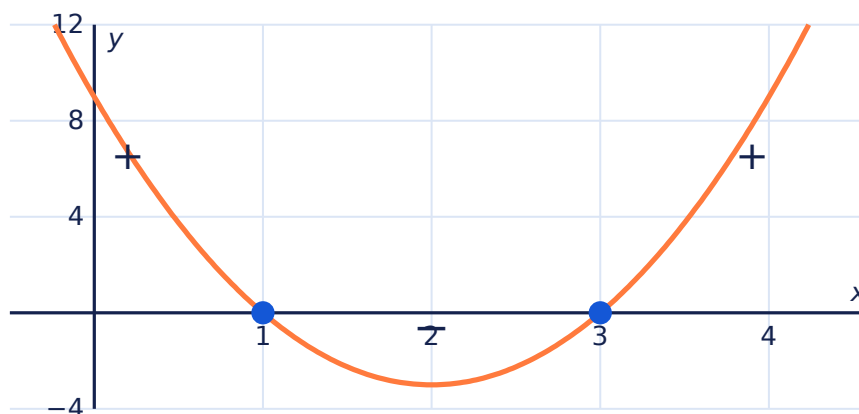
## How does the first derivative describe a function?

### First Derivative Test

A critical number  $c$  is an input where the slope is zero or fails to exist. If  $f'(x)$  changes from positive to negative at  $c$ ,  $f$  has a local maximum. If it changes from negative to positive,  $f$  has a local minimum.



$$\text{For } f(x) = x^3 - 6x^2 + 9x + 1, f'(x) = 3(x - 1)(x - 3).$$



The sign of  $f'(x)$  tells where  $f$  rises (+) and falls (-).

## How does the second derivative describe concavity?

### Concavity and inflection

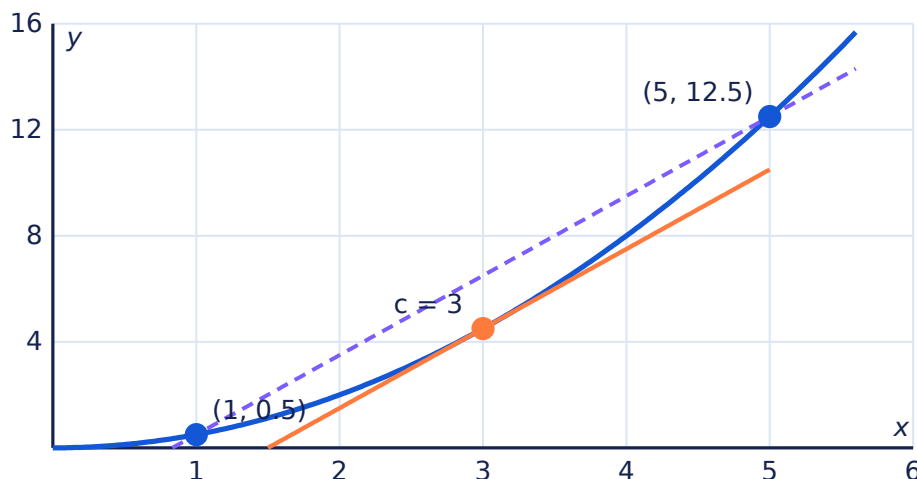
If  $f'' > 0$  on an interval, the graph is concave up. If  $f'' < 0$ , it is concave down. An inflection point is where  $f''$  changes sign. If  $f'(c) = 0$  and  $f''(c) > 0$ , there is a local minimum; if  $f''(c) < 0$ , a local maximum.

Here  $f''(x) = 6x - 12$ , which changes sign at  $x = 2$ , so the inflection point is  $(2, 3)$ .

## What do the Mean Value and Extreme Value Theorems guarantee?

### Two theorems

**Extreme Value Theorem:** when a function is unbroken over a closed span  $[a, b]$ , it reaches a highest and a lowest value there. **Mean Value Theorem:** when  $f$  has no breaks on  $[a, b]$  and a slope everywhere inside, then at some point  $c$  between  $a$  and  $b$  the tangent slope  $f'(c)$  equals the secant slope  $\frac{f(b) - f(a)}{b - a}$ .



For  $f(x) = x^2/2$  on  $[1, 5]$  the secant slope is 3, and the tangent at  $x = 3$  has the same slope.

### Candidates Test for absolute extrema

1. List every critical number lying inside the interval.
2. Compute  $f$  at those numbers and at the two ends of the interval.
3. Compare the outputs: the biggest is the absolute maximum, the smallest is the absolute minimum.

### Common mistake

Setting  $f'(x) = 0$  finds candidates only. A zero derivative does not guarantee an extremum, as  $x^3$  shows at  $x = 0$ .

### Mathbot's tip

For optimization, write the quantity to maximize, use the constraint to leave one variable, then run the Candidates Test.

### Key points to remember

- A critical number is where  $f'$  is zero or undefined.
- The First Derivative Test uses sign changes of  $f'$ ; the Second Derivative Test uses the sign of  $f''$  at a critical number.
- An inflection point is where  $f''$  changes sign, not merely where  $f'' = 0$ .
- Absolute extrema on a closed interval come from the Candidates Test.
- The Mean Value Theorem needs continuity on and differentiability on  $(a, b)$ .

## Using Derivatives to Analyze Graphs and Optimize: frequently asked questions

### What is the difference between a critical point and an extremum?

A critical number is only a candidate where the derivative is zero or undefined. It becomes a local maximum or minimum only if the derivative changes sign there.

### When does the Second Derivative Test fail?

It fails when  $f''(c) = 0$  or does not exist. Then use the First Derivative Test, because the point could be a minimum, maximum or neither.

### How do I find absolute extrema on a closed interval?

Use the Candidates Test: evaluate the function at every critical number inside the interval and at both endpoints, then compare. The largest and smallest values are the absolute extrema.

### Why does the Mean Value Theorem need differentiability?

Without it, the graph can have a corner. For  $|x - 2|$  on  $[0, 5]$  the secant slope is  $1/5$ , but the slopes of the graph are only  $-1$  and  $1$ , so no point matches.

### **How do I set up an optimization problem?**

Draw a picture, name the variables, write the quantity to optimize, use the constraint to reduce it to one variable, then find critical numbers and test the endpoints.