

Using Derivatives to Analyze Graphs and Optimize

Grade 12 – AP Calculus AB math · Worksheet Answer Key · Standards: AP Calculus AB Unit 5

Each solution lists the derivative, the sign analysis or candidate values, and the justification. If an answer differs, check that every endpoint was tested and that the sign of the derivative was checked on both sides of each critical number.

Standards: AP Calculus AB Unit 5 · **Level:** Grade 12 AP Calculus AB

Problem 1 Absolute extrema on a closed interval

$f(x) = 3x^2 - 6x - 9 = 3(x - 3)(x + 1)$, so the critical numbers are -1 and 3 . Candidates: $f(-2) = 3$, $f(-1) = 10$, $f(3) = -22$, $f(4) = -15$. The absolute maximum is **10 at $x = -1$** and the absolute minimum is **-22 at $x = 3$** .

Problem 2 Increasing and decreasing intervals

$g'(x) = 4x^3 - 16x = 4x(x - 2)(x + 2)$, with zeros at -2 , 0 and 2 . Test signs: negative for $x < -2$, positive on $(-2, 0)$, negative on $(0, 2)$, positive for $x > 2$. So g is **increasing on $(-2, 0)$ and $(2, \infty)$** and **decreasing on $(-\infty, -2)$ and $(0, 2)$** .

Problem 3 First Derivative Test with an exponential

$h'(x) = e^{-2x} - 2xe^{-2x} = (1 - 2x)e^{-2x}$. The factor e^{-2x} is always positive, so $h' = 0$ at $x = 1/2$. The sign is positive before $1/2$ and negative after, so there is a **local maximum at $x = 1/2$** with value **$1/(2e)$** .

Problem 4 Concavity and inflection points

$f''(x) = 12x^2 - 12 = 12(x - 1)(x + 1)$, with zeros at ± 1 . It is positive for $x < -1$ and $x > 1$ and negative on $(-1, 1)$. So f is **concave up on $(-\infty, -1)$ and $(1, \infty)$** , **concave down on $(-1, 1)$** , with inflection points at **$x = -1$ and $x = 1$** .

Problem 5 Second Derivative Test

$f(x) = 3x^2 - 12$, so the critical numbers are ± 2 . $f''(x) = 6x$. At $x = 2$, $f'' = 12 > 0$: **local**

minimum with $f(2) = -9$. At $x = -2$, $f' = -12 < 0$: **local maximum** with $f(-2) = 23$.

Problem 6 Mean Value Theorem

f is a polynomial, so it is continuous on $[1, 7]$ and differentiable on $(1, 7)$. The average rate is $\frac{f(7) - f(1)}{7 - 1} = (8 - (-4))/6 = 2$. Solve $f'(c) = 2c - 6 = 2$, so $c = 4$, which lies in $(1, 7)$.

Problem 7 When the Mean Value Theorem does not apply

f is continuous but not differentiable at $x = 2$, which is inside $(0, 5)$, so a hypothesis fails. The average rate is $(3 - 2)/5 = 1/5$. But $f'(x)$ equals -1 for $x < 2$ and $+1$ for $x > 2$, and never $1/5$. **No such c exists**, which shows that the hypotheses cannot be dropped.

Problem 8 Locating an inflection point

$f''(x) = 12x - 30$, which is negative for $x < 5/2$ and positive for $x > 5/2$. The sign changes, so the inflection point has $x = 5/2$ and $f(5/2) = 31.25 - 93.75 + 90 = 27.5$. The point is **(5/2, 55/2)**.

Problem 9 Reading the graph of a derivative

f increases where $f'(x) > 0$: on **$(-2, 1)$ and $(4, \infty)$** . It decreases on $(-\infty, -2)$ and $(1, 4)$. The sign of $f'(x)$ changes from negative to positive at $x = -2$ and $x = 4$, giving **local minima** there, and from positive to negative at $x = 1$, giving a **local maximum**.

Problem 10 Inflection points from a derivative formula

Expand: $f'(x) = (x^3 - 3x^2 - 6x + 8)/4$. Then $f''(x) = (3x^2 - 6x - 6)/4$. Setting it to zero gives $x = 1 \pm \sqrt{3}$. The sign of f'' changes at both roots (it is a parabola opening upward), so the inflection points are at **$x = 1 - \sqrt{3} \approx -0.73$ and $x = 1 + \sqrt{3} \approx 2.73$** .

Problem 11 Absolute extrema with a reciprocal

$f(x) = 1 - 4/x^2 = 0$ gives $x = 2$ (only positive root). Candidates: $f(1) = 5$, $f(2) = 4$, $f(5) = 5.8$. The absolute minimum is **4 at $x = 2$** and the absolute maximum is **$29/5 = 5.8$ at $x = 5$** .

Problem 12 Fencing a pen against a barn

With two sides x and one side y , $2x + y = 180$, so $y = 180 - 2x$. Area $A = x(180 - 2x)$ for $0 < x < 90$. $A' = 180 - 4x = 0$ gives $x = 45$. Since $A'' = -4 < 0$, this is a maximum. Then $y = 90$. The dimensions are **45 ft by 90 ft** with maximum area **4,050 ft²**.

Problem 13 Open box from cardboard

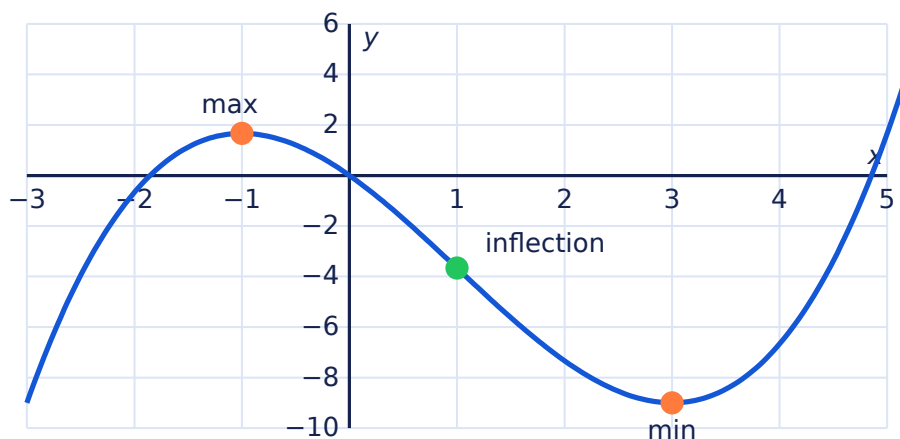
The volume is $V = x(30 - 2x)(20 - 2x) = 4x^3 - 100x^2 + 600x$ with $0 < x < 10$. $V' = 12x^2 - 200x + 600 = 0$, or $3x^2 - 50x + 150 = 0$. The quadratic formula gives $x = (50 \pm \sqrt{700})/6$, so $x \approx 3.92$ or 12.74 . Only 3.92 is in $(0, 10)$. At the endpoints $V = 0$, so **$x \approx 3.92$ in** gives the maximum volume $V \approx \mathbf{1,056 \text{ in}^3}$.

Problem 14 Max and min on an ellipse

Differentiate: $2x + 8y \frac{dy}{dx} = 0$, so $\frac{dy}{dx} = -x/(4y)$. It is zero when $x = 0$, giving $y = \pm 2$. Second derivative: $\frac{d^2y}{dx^2} = -\frac{1}{4} \cdot (y - x \frac{dy}{dx}) / y^2$, which equals $-1/(4y)$ at $x = 0$. At $(0, 2)$: $-1/8 < 0$, a **local maximum**. At $(0, -2)$: $+1/8 > 0$, a **local minimum**.

Problem 15 Connecting f , f' and f''

The sign of $f'(x)$ changes from $+$ to $-$ at $x = -1$: a **local maximum**. It changes from $-$ to $+$ at $x = 3$: a **local minimum**. The concavity changes at $x = 1$: an **inflection point**. The graph rises while concave down, peaks at -1 , falls (concave down until 1 , then concave up), bottoms at 3 and rises again, concave up.



A function matching the description: $f(x) = x^3/3 - x^2 - 3x$.

Problem 16 Ticket price and revenue

Revenue $R(p) = p(500 - 20p)$, with $R'(p) = 500 - 40p = 0$ at $p = 12.5$. Candidates: $R(5) = 2,000$, $R(12.5) = 3,125$, $R(20) = 2,000$. The maximum revenue is **\$3,125 at a price of \$12.50**.

Problem 17 Minimizing the surface of a can

From $V = \pi r^2 h = 500$, $h = 500/(\pi r^2)$. The surface area is $S = 2\pi r^2 + 1000/r$. $S' = 4\pi r - 1000/r^2 = 0$ gives $r^3 = 250/\pi$, so $r \approx \mathbf{4.3 \text{ cm}}$. Then $h = 500/(\pi \cdot 4.3^2) \approx \mathbf{8.6 \text{ cm}}$ (twice the radius). Since $S'' >$

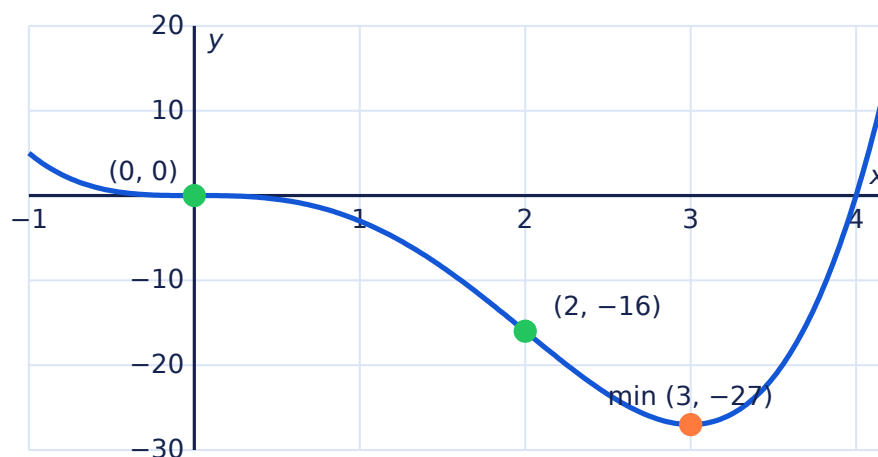
0, it is a minimum.

Problem 18 Mean Value Theorem for a drone

The average rate is $(95 - 15)/8 = 10$ m/s. By the Mean Value Theorem, **some time in $(0, 8)$ has $h' = 10$** . For 12 m/s the theorem gives no guarantee, and a drone with constant speed 10 m/s never reaches 12, so **12 m/s is not guaranteed**.

Problem 19 Complete analysis of a quartic

$f(x) = 4x^3 - 12x^2 = 4x^2(x - 3)$, so the critical numbers are 0 and 3. $f(x) < 0$ on both sides of 0 and changes from $-$ to $+$ at 3: **decreasing on $(-\infty, 3)$, increasing on $(3, \infty)$, local minimum $f(3) = -27$** , and no extremum at 0. $f'' = 12x^2 - 24x = 12x(x - 2)$, positive for $x < 0$ and $x > 2$, negative on $(0, 2)$. So **concave up on $(-\infty, 0)$ and $(2, \infty)$, concave down on $(0, 2)$** , with inflection points **$(0, 0)$ and $(2, -16)$** .



Key points of the quartic.

Problem 20 Finding coefficients from conditions

$f'(x) = 6x + 2a = 0$ at $x = 2$ gives $a = -6$. $f(-1) = 3 - 2a + b = 3 + 12 + b = 0$, so $b = -15$. $f(0) = c = -7$. So $f(x) = x^3 - 6x^2 - 15x - 7$ and $f'(x) = 3(x - 5)(x + 1)$. Then $f'(-1) = -18 < 0$, so **local maximum at $x = -1$** ($f(-1) = 1$), and $f'(5) = 18 > 0$, so **local minimum at $x = 5$** ($f(5) = -107$). The constants are **$a = -6$, $b = -15$, $c = -7$** .