

Applications of Integration: Area and Volume

Grade 12 – AP Calculus AB math · Lesson · Standards: AP Calculus AB Unit 8

Quick answer: Integrals measure accumulated quantities. The average value of a function is its integral divided by the interval length, the area between curves is the integral of top minus bottom, and volumes come from integrating the area of discs, washers or known cross sections along an axis.

Standards: AP Calculus AB Unit 8 · **Level:** Grade 12 AP Calculus AB

What is the average value of a function?

Average value

Divide the integral by the interval length: the mean value of f over $[a, b]$ is $\frac{(\int_a^b f(x) dx)}{(b-a)}$. Picture a flat roof over $[a, b]$ that covers exactly as much area as the curve does.



For $f(x) = x^2$ on $[0, 3]$, the area is 9, so the average value is $9/3 = 3$.

How do you find the area between two curves?

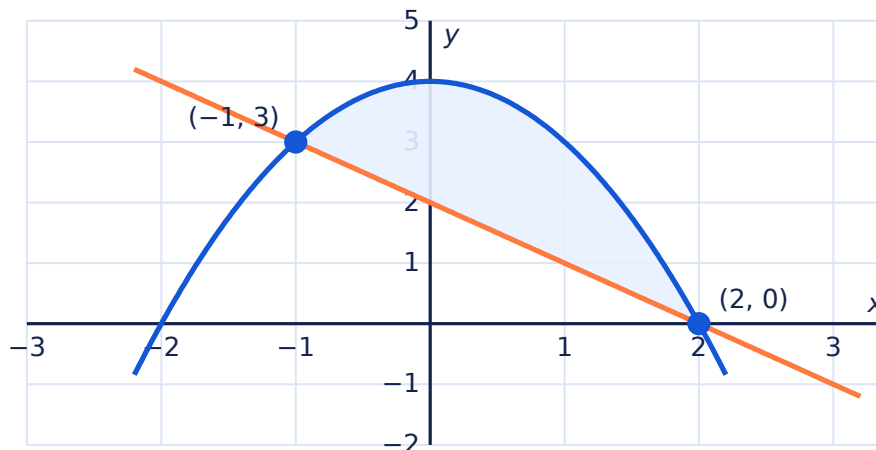
Area between curves

1. Find where the curves intersect to get the limits.
2. Decide which curve is on top (or on the right if you integrate in y).
3. Integrate (top – bottom) dx , or (right – left) dy .

Worked example

The curves $y = 4 - x^2$ and $y = 2 - x$ meet where $x^2 - x - 2 = 0$, so $x = -1$ and $x = 2$. The area

is $\int_{-1}^2 (2 + x - x^2) dx = [2x + x^2/2 - x^3/3] \text{ from } -1 \text{ to } 2 = \mathbf{9/2}$.

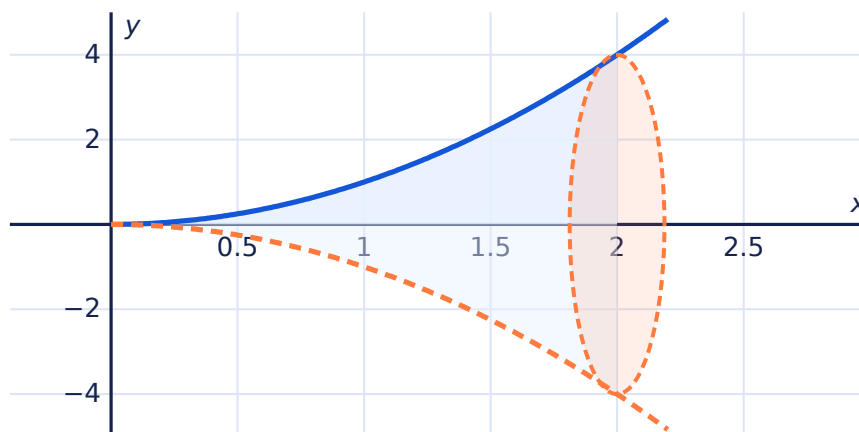


The parabola lies above the line between the intersection points.

How do you find volumes by discs, washers and cross sections?

Volume formulas

Disc: $V = \pi \int_a^b (R(x))^2 dx$. **Washer:** $V = \pi \int_a^b [(R)^2 - (r)^2] dx$. **Cross sections:** $V = \int_a^b A(x) dx$, where A is the area of a slice, such as a square of side s or a semicircle of diameter s .



Revolving $y = x^2$ on $[0, 2]$ about the x -axis gives $V = \pi \int_0^2 x^4 dx = 32\pi/5$.

When the axis is a line like $y = -1$, measure each radius from that line: the outer radius is the distance from the axis to the far curve, and the inner radius is the distance to the near curve.

Common mistake

Squaring the difference, $(\text{outer} - \text{inner})^2$, is wrong for washers. Square each radius first, then subtract.

Mathbot's tip

Draw the region and one typical slice before writing any integral. The picture tells you the limits and the radius.

Key points to remember

- Average value of f on is the integral of f divided by $b - a$.
- Area between curves is the integral of (top – bottom) or (right – left).
- Disc volume: π times the integral of R squared; washers subtract inner squared from outer squared.
- Cross-section volume integrates the slice area $A(x)$, such as s^2 for squares.
- Net change equals the integral of the rate; total distance integrates $|v|$.

Applications of Integration: Area and Volume: frequently asked questions

What is the difference between displacement and total distance?

Displacement is the integral of velocity and can be negative. Total distance is the integral of the absolute value of velocity, found by splitting the interval at every time when the velocity is zero.

How do I decide between discs and washers?

Use discs when the region touches the axis of revolution along its whole length. Use washers when a gap separates the region from the axis, producing a hole.

What if the curves cross more than twice?

Find every intersection, then split the area into separate pieces. On each piece decide which curve is on top and integrate top minus bottom.

How do I find the volume of a solid with square cross sections?

Find the side length s of the square at each x from the base region, square it to get $A(x) = s^2$, and integrate over the interval.

How do accumulation problems with rates in and out work?

The amount at time t equals the starting amount plus the integral of (rate in – rate out). The amount is greatest or least where the net rate changes sign.