

Applications of Integration: Area and Volume

Grade 12 – AP Calculus AB math · Worksheet Answer Key · Standards: AP Calculus AB Unit 8

Each solution identifies the limits, the top and bottom or the radii, and then evaluates the integral. If your answer differs, check which curve is on top and whether the radii were measured from the correct axis of revolution.

Standards: AP Calculus AB Unit 8 · **Level:** Grade 12 AP Calculus AB

Problem 1 Average value of a polynomial

$\int_2^5 (4x^2 - 3x) dx = [4x^3/3 - 3x^2/2]$ from 2 to 5 = $(500/3 - 75/2) - (32/3 - 6) = 124.5$. Divide by the width 3: **$83/2 = 41.5$** .

Problem 2 Average value of sine

$\int_0^\pi \sin x dx = [-\cos x]$ from 0 to $\pi = 1 - (-1) = 2$. The average value is $2/\pi$, or **$2/\pi \approx 0.64$** .

Problem 3 Position from velocity

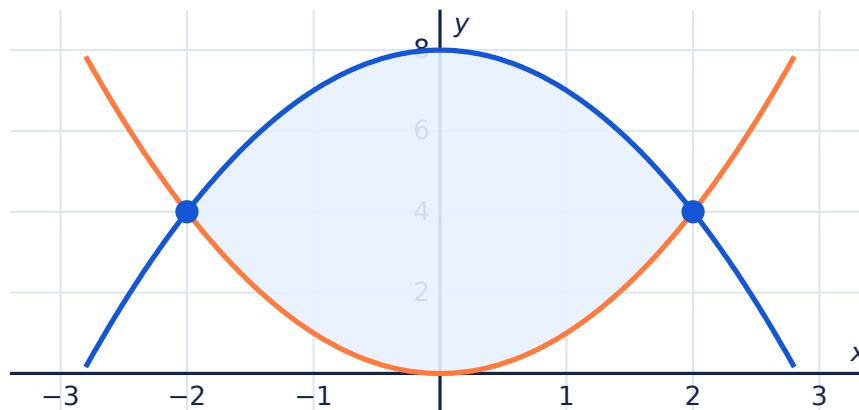
$s(3) = s(0) + \int_0^3 (4t - t^2) dt = 5 + [2t^2 - t^3/3]$ from 0 to 3 = $5 + (18 - 9) =$ **14 meters**.

Problem 4 Net change and total distance

An antiderivative is $s(t) = t^3/3 - 7t^2/2 + 10t$. Net change: $s(6) - s(0) = 72 - 126 + 60 =$ **6 m**. The velocity $(t - 2)(t - 5)$ is zero at 2 and 5. Positions: $s(2) = 26/3$, $s(5) = 25/6$, $s(6) = 6$. Path length: $26/3 + (26/3 - 25/6) + (6 - 25/6) = 52/6 + 27/6 + 11/6 =$ **15 m**.

Problem 5 Area between two parabolas

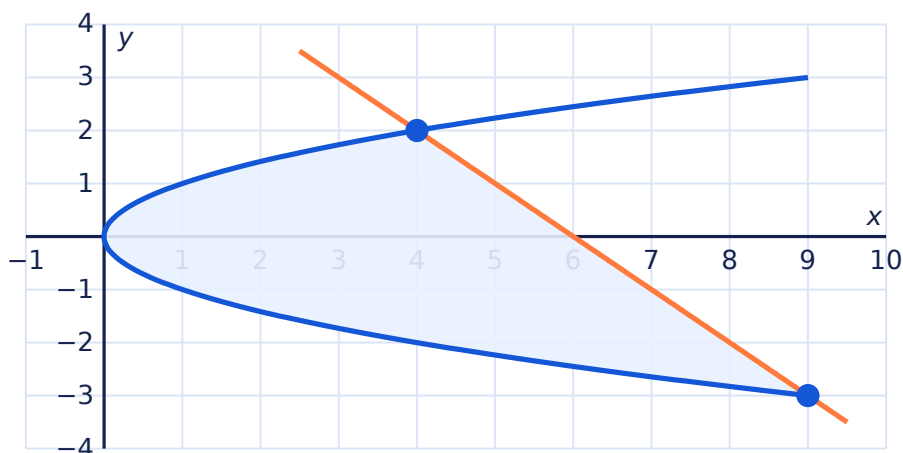
Set $8 - x^2 = x^2$: $x = \pm 2$. On $[-2, 2]$ the downward parabola is on top. Area = $\int_{-2}^2 (8 - 2x^2) dx = [8x - 2x^3/3]$ from -2 to $2 = 2(16 - 16/3) =$ **$64/3$** .



The enclosed region.

Problem 6 Area integrated with respect to y

Set $y^2 = 6 - y$: $y^2 + y - 6 = 0$, so $y = 2$ or $y = -3$. The line is on the right. Area = $\int_{-3}^2 (6 - y - y^2) dy = [6y - y^2/2 - y^3/3]$ from -3 to $2 = (22/3) - (-27/2) = \mathbf{125/6}$.



The shaded region.

Problem 7 Square cross sections

The curves meet at $x = 0$ and $x = 2$, with the line above. The edge is $2x - x^2$, so $A(x) = (2x - x^2)^2 = 4x^2 - 4x^3 + x^4$. Then $V = \int_0^2 (4x^2 - 4x^3 + x^4) dx = [4x^3/3 - x^4 + x^5/5]$ from 0 to $2 = 32/3 - 16 + 32/5 = \mathbf{16/15}$.

Problem 8 Semicircular cross sections

The diameter is $\sqrt{x}$, so the radius is $\sqrt{x}/2$ and the slice area is $(\pi/2)(x/4) = \pi x/8$. Then $V = \int_0^9 \pi x/8 dx = (\pi/8)(81/2) = \mathbf{81\pi/16}$.

Problem 9 Disc method

Disc radius $\sqrt{x+1}$, so $V = \pi \int_0^3 (x+1) dx = \pi[x^2/2 + x]$ from 0 to 3 = $\pi(9/2 + 3) = 15\pi/2$.

Problem 10 Washer method about the x-axis

The curves meet at $x = 0$ and $x = 1$, with $\sqrt{x}$ above, so the outer radius is $\sqrt{x}$ and the inner radius is x^2 . $V = \pi \int_0^1 (x - x^4) dx = \pi(1/2 - 1/5) = 3\pi/10$.

Problem 11 Revolving about a horizontal line

Measure from the line $y = -1$: the outer radius is $x^2 + 1$ and the inner radius is $0 + 1 = 1$. $V = \pi \int_0^1 [(x^2 + 1)^2 - 1^2] dx = \pi \int_0^1 (x^4 + 2x^2) dx = \pi(1/5 + 2/3) = 13\pi/15$.

Problem 12 Revolving about the y-axis

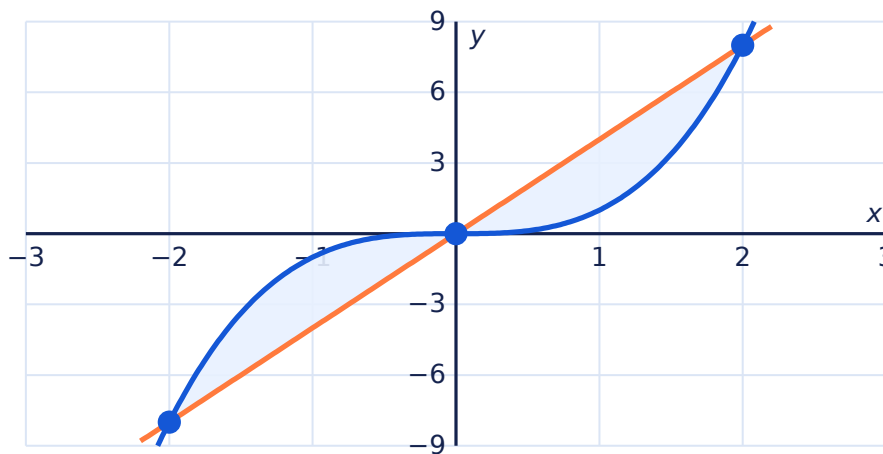
Solve for x : $x = \sqrt{y}$, so the disc radius is $\sqrt{y}$. $V = \pi \int_0^4 y dy = \pi(16/2) = 8\pi$.

Problem 13 Tank with water flowing in and out

The amount is $A(t) = 120 + \int_0^t (2s + 4 - 10) ds = 120 + t^2 - 6t$. So $A(10) = 120 + 100 - 60 = 160$ gallons. The net rate $2t - 6$ changes from negative to positive at $t = 3$, so the least amount is at $t = 3$: $A(3) = 111$ gallons.

Problem 14 Area with three intersection points

Set $x^3 = 4x$: $x = -2, 0, 2$. On $[0, 2]$ the line is on top; on $[-2, 0]$ the cubic is on top, and by symmetry the two areas are equal. Area = $2 \int_0^2 (4x - x^3) dx = 2[2x^2 - x^4/4]$ from 0 to 2 = $2(8 - 4) = 8$.



Problem 15 Area where curves cross

The curves cross at $x = \pi/4$. On $[0, \pi/4]$, $\cos$ is above $\sin$; on $[\pi/4, \pi/2]$, $\sin$ is above $\cos$. Area = $\int_0^{\pi/4} (\cos x - \sin x) dx + \int_{\pi/4}^{\pi/2} (\sin x - \cos x) dx$. Each piece equals $\sqrt{2} - 1$. The total is $2\sqrt{2} - 2 \approx 0.83$.

Problem 16 Revolving about a vertical line

Since $y = \sqrt{x}$ means $x = y^2$, at height y the region runs from $x = y^2$ to $x = 4$, so the disc radius (the distance to the axis $x = 4$) is $4 - y^2$, and y runs from 0 to 2. $V = \pi \int_0^2 (4 - y^2)^2 dy = \pi \int_0^2 (16 - 8y^2 + y^4) dy = \pi [16y - 8y^3/3 + y^5/5]$ from 0 to 2 = $\pi(32 - 64/3 + 32/5) = 256\pi/15$.

Problem 17 Greenhouse temperature

Mean = $(1/12) \int_0^{12} (58 + 12 \sin(\pi h/12)) dh$. The integral of $\sin(\pi h/12)$ over $[0, 12]$ is $[-(12/\pi)\cos(\pi h/12)] = 24/\pi$. So the mean is $58 + (12/12)(24/\pi) = 58 + 24/\pi \approx 65.6^\circ\text{F}$.

Problem 18 Volume of a tent

A leg is $9 - x^2$, so the triangle has area $(1/2)(9 - x^2)^2$. $V = (1/2) \int_{-3}^3 (81 - 18x^2 + x^4) dx = \int_0^3 (81 - 18x^2 + x^4) dx = [81x - 6x^3 + x^5/5]$ from 0 to 3 = $243 - 162 + 48.6 = 129.6 \text{ ft}^3$ (648/5).

Problem 19 Area and volume from one region

Area = $\int_1^e (1/x) dx = \ln e - \ln 1 = 1$. Volume = $\pi \int_1^e (1/x^2) dx = \pi [-1/x]$ from 1 to $e = \pi(1 - 1/e) \approx 1.99$.

Problem 20 Mean Value Theorem for integrals

$\int_1^4 (x^2 + 1) dx = [x^3/3 + x]$ from 1 to 4 = $(64/3 + 4) - (1/3 + 1) = 24$. The average value is $24/3 = 8$. Solve $c^2 + 1 = 8$: $c = \sqrt{7} \approx 2.65$, which lies in $[1, 4]$.