

Chain Rule and Implicit Differentiation

Grade 12 – AP Calculus AB math · Worksheet Answer Key · Standards: AP Calculus AB Unit 3

Each solution names the outer and inner layers or the implicit step used. When your answer differs, check whether the inner derivative or a factor of dy/dx was left out, since those are the most common slips in this unit.

Standards: AP Calculus AB Unit 3 · **Level:** Grade 12 AP Calculus AB

Problem 1 Chain rule with a power

Outer: $4(\cdot)^3$. Inner derivative: 6. So $f'(x) = 4(6x + 1)^3 \cdot 6 = \mathbf{24(6x + 1)^3}$.

Problem 2 Chain rule with a root

Write $f(x) = (3x^2 + 16)^{1/2}$. Then $f'(x) = \frac{1}{2}(3x^2 + 16)^{-1/2} \cdot 6x = \frac{3x}{\sqrt{3x^2 + 16}}$.

Problem 3 Chain rule with e

The derivative of e^u is e^u times u' . So $f'(x) = \mathbf{(2x - 3)e^{x^2 - 3x}}$.

Problem 4 Chain rule with a logarithm

The derivative of $\ln u$ is u'/u . So $f'(x) = \frac{(2x + 4)}{(x^2 + 4x)}$.

Problem 5 Chain rule with sine

Outer: $\cos(3x^2)$. Inner derivative: $6x$. So $f'(x) = \mathbf{6x \cos(3x^2)}$.

Problem 6 Chain rule at a point

$f(x) = 3(2x^2 - 1)^2 \cdot 4x$. At $x = 1$: $3(1)^2 \cdot 4 = \mathbf{12}$.

Problem 7 Chain rule from a table

The chain rule gives $h'(1) = f(g(1)) \cdot g'(1) = f(2) \cdot g'(1) = (-3)(4) = \mathbf{-12}$.

Problem 8 Product rule with the chain rule

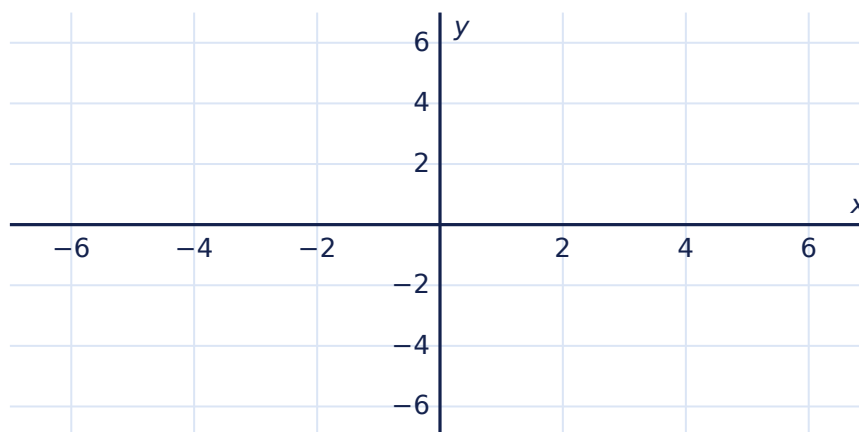
Product rule: $3x^2e^{-2x} + x^3 \cdot (-2)e^{-2x}$. Factor: $f(x) = e^{-2x}(3x^2 - 2x^3)$.

Problem 9 Quotient rule with the chain rule

Quotient rule with $(\sin 2x)' = 2 \cos 2x$: $f(x) = \frac{(2 \cos 2x \cdot x^2 - \sin 2x \cdot 2x)}{x^4} = \frac{(2x \cos 2x - 2 \sin 2x)}{x^3}$.

Problem 10 Slope on a circle

Differentiate: $2x + 2y \frac{dy}{dx} = 0$, so $\frac{dy}{dx} = -\frac{x}{y}$. At (5, 12) the slope is **-5/12**.



The tangent at (5, 12).

Problem 11 Implicit differentiation with a product

Differentiate: $2xy + x^2 \frac{dy}{dx} + 3y^2 \frac{dy}{dx} = 0$. Solve: $\frac{dy}{dx} = \frac{(-2xy)}{(x^2 + 3y^2)}$. At (1, 2): $-4/(1 + 12) = \mathbf{-4/13}$.

Problem 12 Tangent line to an implicit curve

Differentiate: $2x + y + x \frac{dy}{dx} + 2y \frac{dy}{dx} = 0$. So $\frac{dy}{dx} = \frac{-(2x + y)}{(x + 2y)}$. At (2, 1): $-5/4$. The tangent line is $y - 1 = -5/4(x - 2)$, or **$y = -5/4 x + 7/2$** .

Problem 13 Derivative of an inverse function

Find the input with $f(x) = 10$: $f(2) = 8 + 2 = 10$. Also $f(x) = 3x^2 + 1$, so $f(2) = 13$. Then $g'(10) = 1/f'(2) = \mathbf{1/13}$.

Problem 14 Inverse derivative from a table

Since $f(2) = 7$, $g(7) = 2$. So $g'(7) = 1/f(2) = \mathbf{1/3}$.

Problem 15 Inverse trigonometric derivatives

(a) The derivative of $\arctan u$ is $u'/(1 + u^2)$. So the answer is $4/(1 + 16x^2)$.

(b) The derivative of $\arcsin u$ is $u'/\sqrt{1 - u^2}$. So the answer is $2x/\sqrt{1 - x^4}$.

Problem 16 Second derivative of a sum

$f(x) = 5x^4 - 12x^2 + 2e^{2x}$. Then $f'(x) = 20x^3 - 24x + 4e^{2x}$. At 0: $0 - 0 + 4 = \mathbf{4}$.

Problem 17 Second derivative of an implicit curve

First, $dy/dx = -x/y$. Then $d^2y/dx^2 = -(y - xdy/dx)/y^2 = -(y + x^2/y)/y^2 = -(x^2 + y^2)/y^3 = -36/216$. At (0, 6): $-36/216 = \mathbf{-1/6}$.
The curve is concave down there.

Problem 18 Pressure on a climbing balloon

By the chain rule, $dP/dt = P'(h) \cdot h'(t) = -20e^{-0.2h} \cdot 3 = -60e^{-0.2h}$. At $t = 2$, $h = 11$, so $dP/dt = -60e^{-2.2} \approx \mathbf{-6.65 \text{ kPa per hour}}$.

Problem 19 Camera tracking a rocket

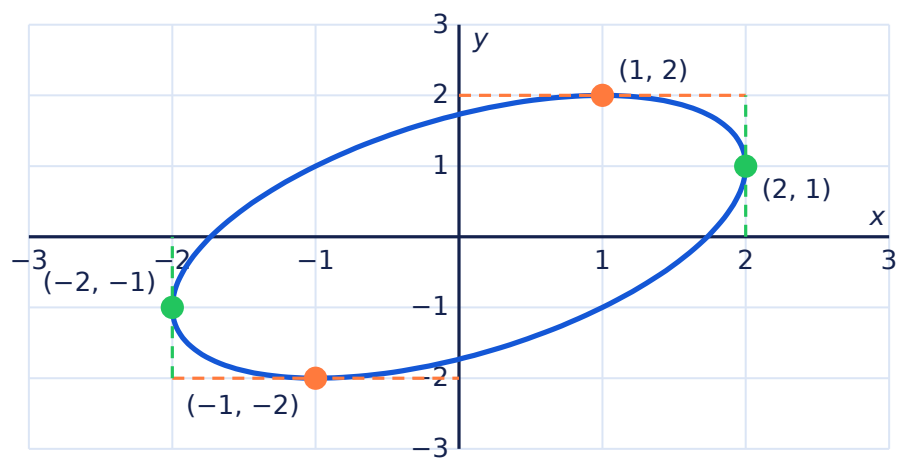
The derivative of $\arctan u$ is $u'/(1 + u^2)$ with $u = x/20$ and $u' = 1/20$. So $d\theta/dx = \frac{1/20}{(1 + x^2/400)} = \frac{20}{(400 + x^2)}$. At $x = 15$: $20/625 = \mathbf{4/125 \approx 0.032 \text{ radian per foot}}$.

Problem 20 Horizontal and vertical tangents

Differentiate: $2x - y - x^2 dy/dx + 2y dy/dx = 0$, so $dy/dx = (y - 2x)/(2y - x)$.

Horizontal: $y = 2x$. Substitute: $x^2 - 2x^2 + 4x^2 = 3x^2 = 3$, so $x = \pm 1$ and the points are **(1, 2) and (-1, -2)**.

Vertical: $x = 2y$. Substitute: $4y^2 - 2y^2 + y^2 = 3y^2 = 3$, so $y = \pm 1$ and the points are **(2, 1) and (-2, -1)**.



Horizontal tangents at the orange points, vertical tangents at the green points.