

Modeling Change with Separable Equations

Grade 12 – AP Calculus AB math · Worksheet Answer Key · Standards: AP Calculus AB Unit 7

Each solution shows the separation, the integration with its constant, and the use of the initial condition. If your answer differs, check that the constant was added at the integration step and that the correct sign of a square root was chosen.

Standards: AP Calculus AB Unit 7 · **Level:** Grade 12 AP Calculus AB

Problem 1 Verifying an exponential solution

Differentiate: $y' = 3 \cdot 2e^{2x} = 6e^{2x}$. The right side is $2y = 2 \cdot 3e^{2x} = 6e^{2x}$. Both sides match, so **the function is a solution**.

Problem 2 General and particular solutions

The derivative of $x^2 + C$ is $2x$ for any constant. For $y(1) = 5$: $1 + C = 5$, so $C = 4$. The particular solution is **$y = x^2 + 4$** .

Problem 3 Verifying a second-order solution

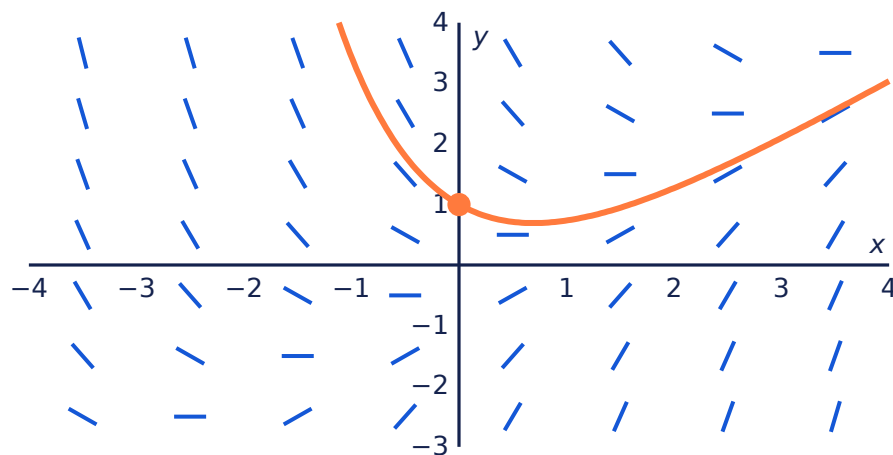
$y' = 10 \cos 2x + 1$ and $y'' = -20 \sin 2x$. Then $y'' + 4y = -20 \sin 2x + 20 \sin 2x + 4x = 4x$. **The equation is satisfied.**

Problem 4 Computing slopes from an equation

Substitute each point: at $(1, 2)$ the slope is $1 + 2 = \mathbf{3}$; at $(-2, 1)$ it is $-2 + 1 = \mathbf{-1}$; at $(0, 0)$ it is **0**.

Problem 5 Reading a slope field

The slope is zero when $x - y = 0$, so along the line **$y = x$** . At $(0, 1)$, $\frac{dy}{dx} = 0 - 1 = -1 < 0$, so the solution is **decreasing** there.



The solution through $(0, 1)$ decreases first, then crosses toward the line $y = x - 1$.

Problem 6 Separable equation with a power

Separate: $dy/y = 4x^3 dx$. Integrate: $\ln|y| = x^4 + C$, so $y = Ae^{x^4}$. With $y(0) = 5$, $A = 5$. The solution is **$y = 5e^{x^4}$** .

Problem 7 Separable equation with a quotient

Separate: $y dy = x dx$. Integrate: $y^2/2 = x^2/2 + C$, so $y^2 = x^2 + K$. With $(2, 3)$: $9 = 4 + K$, so $K = 5$. Since $y(2) > 0$, the solution is **$y = \sqrt{x^2 + 5}$** .

Problem 8 Population growth model

The solution of $dP/dt = kP$ is $P = P_0 e^{kt}$. So $P(t) = 400e^{0.05t}$ and $P(10) = 400e^{0.5} \approx \mathbf{659.5}$ bacteria.

Problem 9 Half-life

The decay constant is $k = -(\ln 2)/8 \approx -0.0866$. The amount is $y = 90e^{-0.0866t}$, equivalently $90(1/2)^{t/8}$. At $t = 20$: $90(1/2)^{2.5} = 90/5.657 \approx \mathbf{15.9 \text{ mg}}$.

Problem 10 Equation with an exponential

Write $e^x - y = e^x/e^y$. Separate: $e^y dy = e^x dx$. Integrate: $e^y = e^x + C$. At $(0, \ln 3)$: $3 = 1 + C$, so $C = 2$. Therefore **$y = \ln(e^x + 2)$** .

Problem 11 Equation with a trigonometric factor

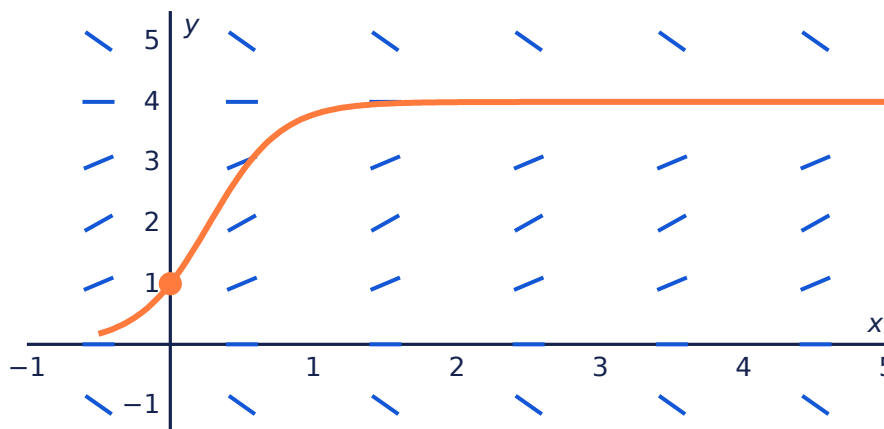
Separate: $dy/y = \sin x dx$. Integrate: $\ln|y| = -\cos x + C$, so $y = Ae^{-\cos x}$. At $x = 0$: $3 = Ae^{-1}$, so $A = 3e$. The solution is **$y = 3e^{1 - \cos x}$** .

Problem 12 Newton's law of cooling

Separate: $dT/(T - 70) = -0.1 dt$. Integrate: $\ln|T - 70| = -0.1t + C$, so $T - 70 = Ae^{-0.1t}$. With $T(0) = 190$, $A = 120$. So $T = 70 + 120e^{-0.1t}$, and $T(10) = 70 + 120e^{-1} \approx 114.1^\circ\text{F}$.

Problem 13 Equilibrium and long-term behavior

Slopes are zero when $y(4 - y) = 0$, so the equilibria are $y = 0$ and $y = 4$. Starting at $(0, 1)$, the slope is positive and the curve rises toward $y = 4$, so the limit is **4**. Differentiating, $\frac{d^2y}{dx^2} = (4 - 2y)\frac{dy}{dx}$, which is positive when $0 < y < 2$. The solution is **concave up until y reaches 2**.



The solution rises toward 4.

Problem 14 Separable equation with a cube root

Separate: $y^2 dy = (x + 1)^2 dx$. Integrate: $y^3/3 = (x + 1)^3/3 + C$, so $y^3 = (x + 1)^3 + K$. At $(0, 2)$: $8 = 1 + K$, so $K = 7$. The solution is $y = \sqrt[3]{(x + 1)^3 + 7}$.

Problem 15 Fish population

With $P = 2,400e^{kt}$, $3,100 = 2,400e^{3k}$, so $k = \ln(31/24)/3 \approx 0.0853$ per year. Then $P(10) = 2,400e^{0.853} \approx 5,633$ fish.

Problem 16 Salt in a tank

Rewrite: $dS/dt = (300 - S)/50$. Separate: $dS/(300 - S) = dt/50$. Integrate: $-\ln|300 - S| = t/50 + C$, so $300 - S = Ae^{-t/50}$. With $S(0) = 0$, $A = 300$. So $S = 300(1 - e^{-t/50})$. At $t = 50$: $300(1 - e^{-1}) \approx 189.6$ lb. As $t \rightarrow \infty$ the amount approaches **300 lb**.

Problem 17 Draining tank

Separate: $h^{-1/2} dh = -0.3 dt$. Integrate: $2\sqrt{h} = -0.3t + C$. At $t = 0$: $2(4) = 8 = C$. So $\sqrt{h} = 4 - 0.15t$ and $h = (4 - 0.15t)^2$. The tank is empty when $4 - 0.15t = 0$, so $t = 4/0.15 \approx \mathbf{26.7}$ minutes.

Problem 18 Choosing the sign of a root

Separate: $y dy = (x + 2) dx$. Integrate: $y^2/2 = x^2/2 + 2x + C$. At $(0, -1)$: $1/2 = C$. Multiply by 2: $y^2 = x^2 + 4x + 1$. Since $y(0) = -1$ is negative, take the negative root: $y = -\sqrt{x^2 + 4x + 1}$.

Problem 19 Equilibrium and exponential behavior

The equilibrium is $y = 3$, where the slope is zero. For $y \neq 3$, separate: $dy/(y - 3) = x dx$. Integrate: $\ln|y - 3| = x^2/2 + C$, so $y - 3 = Ae^{x^2/2}$. At $(0, 5)$: $A = 2$. So $y = 3 + 2e^{x^2/2}$ and $y(2) = 3 + 2e^2 \approx \mathbf{17.8}$.

Problem 20 Time to cool

Separate and integrate: $T - 22 = Ae^{-0.08t}$. With $T(0) = 96$, $A = 74$. So $T = 22 + 74e^{-0.08t}$. Set $T = 50$: $28 = 74e^{-0.08t}$, so $t = \ln(74/28)/0.08 \approx \mathbf{12.1}$ minutes. Because $e^{-0.08t}$ is never zero, T approaches but **never reaches** 22°F .