

Integrals, Riemann Sums and Accumulation

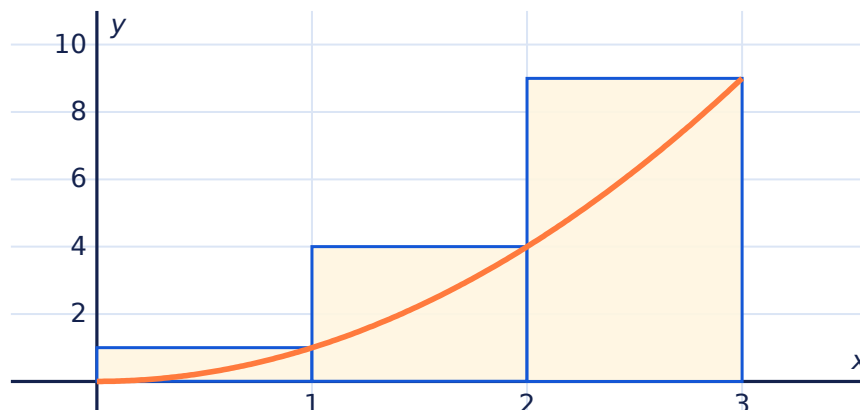
Grade 12 – AP Calculus AB math · Lesson · Standards: AP Calculus AB Unit 6

Quick answer: Integration accumulates change. A definite integral is the limit of Riemann sums and equals net signed area, and the Fundamental Theorem of Calculus links it to antiderivatives so that integrals can be computed by evaluating an antiderivative at the endpoints, with u-substitution handling composite integrands.

Standards: AP Calculus AB Unit 6 · **Level:** Grade 12 AP Calculus AB

What is a Riemann sum?

A Riemann sum approximates the area under a curve by adding up rectangle areas. Left sums use the left endpoint of each subinterval, right sums use the right endpoint, and midpoint sums use the center. For an increasing function, a right sum overestimates and a left sum underestimates.



For $f(x) = x^2$ on $[0, 3]$, the right sum is $1 + 4 + 9 = 14$, which is above the exact area 9.

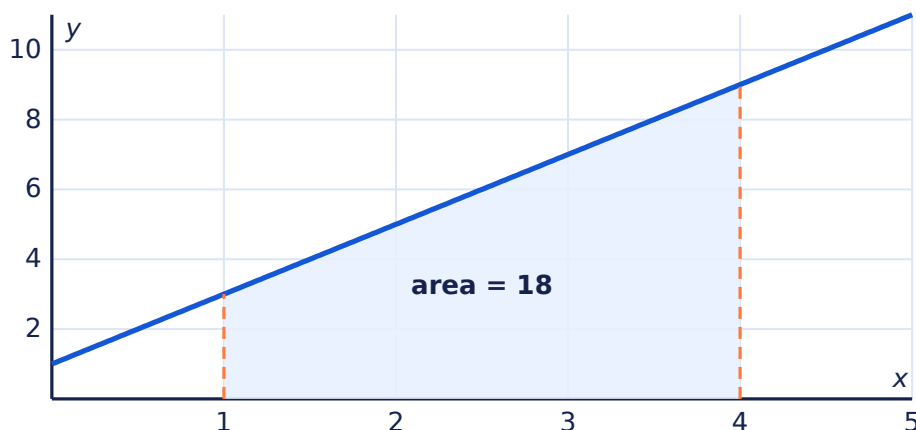
Definite integral

The definite integral $\int_a^b f(x) dx$ is the limit of Riemann sums as the number of rectangles grows. It gives the net signed area between the graph and the x-axis. Area above the axis counts positively and area below counts negatively.

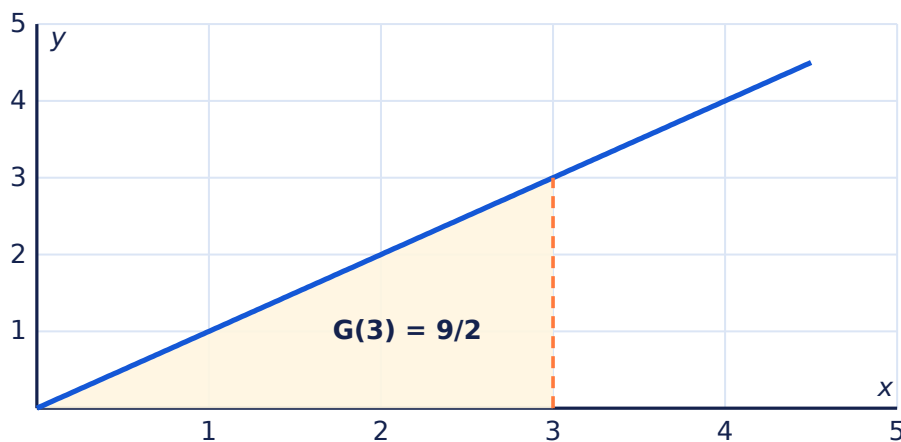
How does the Fundamental Theorem of Calculus connect derivatives and integrals?

Fundamental Theorem of Calculus

Part 1: if $G(x) = \int_a^x f(t) dt$, then $G'(x) = f(x)$. **Part 2:** $\int_a^b f(x) dx = F(b) - F(a)$, where F is any antiderivative of f .



With antiderivative $x^2 + x$, the integral of $2x + 1$ from 1 to 4 is $(16 + 4) - (1 + 1) = 18$.



The accumulated area under $f(t) = t$ from 0 to 3 is $G(3) = 9/2$.

How do you integrate with substitution?

u-substitution

1. Pick u as an inner function whose derivative appears in the integrand.
2. Replace the integrand and dx using $du = u' dx$.
3. Integrate in u , then return to x (or change the limits for a definite integral).

Worked example

For $\int 6x(x^2 + 1)^3 dx$, let $u = x^2 + 1$, so $du = 2x dx$. The integral becomes $\int 3u^3 du = (3/4)u^4 + C = (3/4)(x^2 + 1)^4 + C$.

Common mistake

A definite integral with u-substitution needs new limits in u , or you must convert back to x

before evaluating. Mixing the two gives wrong numbers.

Mathbot's tip

Check any antiderivative by differentiating it. If you get the integrand back, the answer is right.

Key points to remember

- Left, right, midpoint and trapezoidal sums all approximate a definite integral.
- The definite integral gives net signed area: regions below the axis count negatively.
- FTC Part 2: the integral of f from a to b equals $F(b) - F(a)$.
- FTC Part 1: the derivative of the integral of f from a to x is $f(x)$.
- For u -substitution on a definite integral, convert the limits to u .

Integrals, Riemann Sums and Accumulation: frequently asked questions

When is a Riemann sum an overestimate?

For an increasing function, right sums overestimate and left sums underestimate. For a decreasing function the roles reverse. Concavity decides whether trapezoidal sums are over or under.

What does a definite integral represent in context?

It represents the accumulated change of a quantity whose rate is the integrand. If the integrand is a rate in gallons per minute, the integral is in gallons.

How do I differentiate an integral with a variable upper limit?

Use FTC Part 1: substitute the upper limit into the integrand. If the upper limit is a function like x^2 , multiply by its derivative, $2x$.

How do I choose u in a substitution?

Look for an inner function whose derivative, up to a constant factor, appears elsewhere in the integrand. Common choices are the expression inside a power, an exponent or a logarithm.

What is the $+C$ for?

Every antiderivative of a function differs by a constant, so an indefinite integral needs $+C$. Definite integrals do not need it because the constant cancels.