

Integrals, Riemann Sums and Accumulation

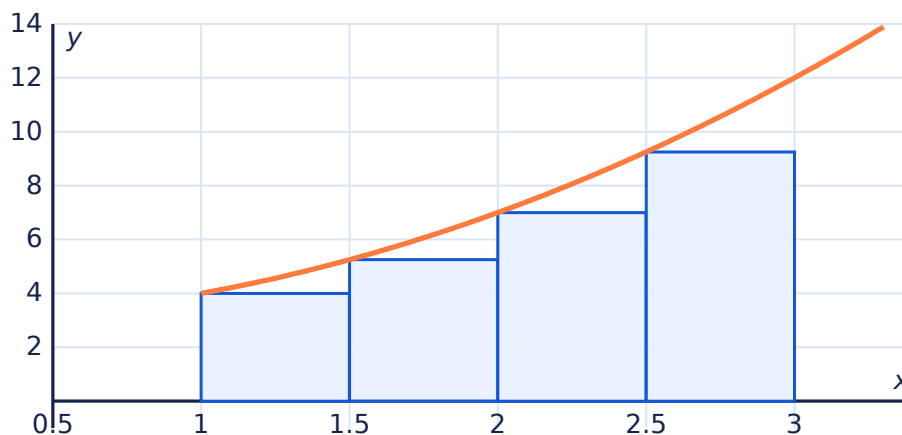
Grade 12 – AP Calculus AB math · Worksheet Answer Key · Standards: AP Calculus AB Unit 6

Each solution shows the antiderivative or table computation and then the evaluation. When your answer differs, check the limits after a substitution and the signs of regions below the axis, which are the most common slips.

Standards: AP Calculus AB Unit 6 · **Level:** Grade 12 AP Calculus AB

Problem 1 Left and right Riemann sums

The width is 0.5. Left sum: $0.5[f(1) + f(1.5) + f(2) + f(2.5)] = 0.5(4 + 5.25 + 7 + 9.25) = 0.5(25.5) = \mathbf{12.75}$. Right sum: $0.5(5.25 + 7 + 9.25 + 12) = 0.5(33.5) = \mathbf{16.75}$. The exact area is $44/3 \approx 14.67$, between the two.



The left rectangles lie under the curve.

Problem 2 Trapezoidal sum from a table

Each trapezoid has width 2. Sum: $(2/2)[5 + 2(8) + 2(9) + 2(7) + 4] = 5 + 16 + 18 + 14 + 4 = \mathbf{57 \text{ gallons}}$.

Problem 3 Antiderivative of a polynomial

Integrate term by term: $2x^3 - 2x^2 + 7x + C$. The answer is $\mathbf{2x^3 - 2x^2 + 7x + C}$.

Problem 4 Antiderivative with ln and exponential

The antiderivatives are $3 \ln|x|$, e^x and $\cos x$ (because the derivative of $\cos x$ is $-\sin x$). The answer is **$3 \ln|x| + e^x + \cos x + C$** .

Problem 5 Meaning of a definite integral

The integral of a rate gives the accumulated change. Between 2 and 6 hours after opening, the truck sells a total of **148 burritos**.

Problem 6 Evaluating with the Fundamental Theorem

An antiderivative of $3x^{1/2}$ is $2x^{3/2}$. So the integral is $2(4)^{3/2} - 2(1)^{3/2} = 2(8) - 2 = \mathbf{14}$.

Problem 7 Trigonometric definite integral

An antiderivative is $2 \sin x + x$. Evaluate: $(2 \sin \pi + \pi) - (2 \sin 0 + 0) = \pi - 0 = \mathbf{\pi}$.

Problem 8 Properties of definite integrals

(a) By additivity, $\int_2^5 = 12 - (-3) = \mathbf{15}$.

(b) Reversing the limits changes the sign: **-12**.

(c) $4(12) - 2(5) = 48 - 10 = \mathbf{38}$.

Problem 9 Substitution with a power

Let $u = x^2 + 5$, so $du = 2x \, dx$. Then $\int u^4 \, du = u^5/5 + C = \mathbf{(x^2 + 5)^5/5 + C}$.

Problem 10 Substitution with an exponential

Let $u = x^3$, so $du = 3x^2 \, dx$ and $x^2 \, dx = du/3$. The integral is $(1/3) \int e^u \, du = \mathbf{(1/3)e^{x^3} + C}$.

Problem 11 Definite integral with substitution

Let $u = x^2 + 9$, so $x \, dx = du/2$. The limits become $u = 9$ when $x = 0$ and $u = 16$ when $x = \sqrt{7}$. Then $(1/2) \int_9^{16} u^{1/2} \, du = (1/2)(2/3)(16^{3/2} - 9^{3/2}) = (1/3)(64 - 27) = \mathbf{37/3}$.

Problem 12 Derivative of an accumulation function

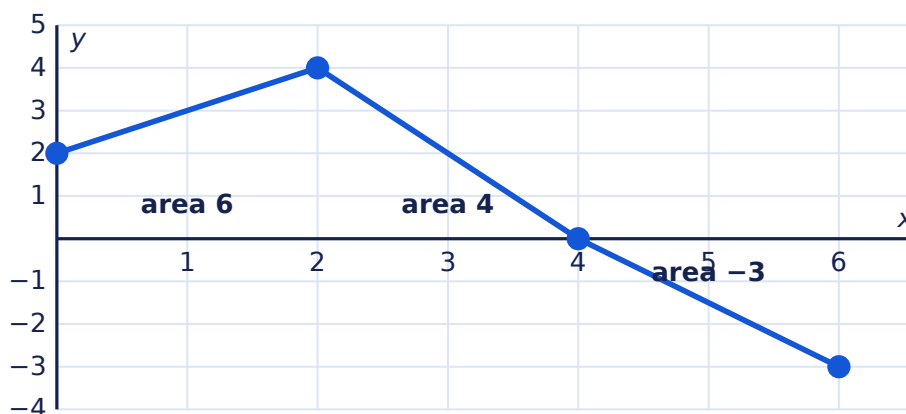
$G(1)$ is an integral over an interval of width zero, so it equals **0**. By FTC Part 1, $G'(x) = \sqrt{(x^3 + 8)}$, so $G'(2) = \sqrt{16} = \mathbf{4}$.

Problem 13 Chain rule with an integral

Apply FTC Part 1 with the chain rule: $H'(x) = (3x^2 + 1) \cdot 2x$. At $x = 2$: $(13)(4) = \mathbf{52}$.

Problem 14 Accumulation from a graph

From 0 to 2 the area under the segment from height 2 to 4 is $(2 + 4)/2 \cdot 2 = 6$, so $g(2) = \mathbf{6}$. From 2 to 4 the area is $(4 + 0)/2 \cdot 2 = 4$, so $g(4) = \mathbf{10}$. From 4 to 6 the region lies below the axis: triangle area $(1/2)(2)(3) = 3$, counted as -3 , so $g(6) = \mathbf{7}$. Because $g' = f$ changes from positive to negative at $x = 4$, the maximum is at $\mathbf{x = 4}$, where $g = 10$.



Signed areas of the three pieces.

Problem 15 Integrating after long division

Divide: $x^2 + 5x + 7 = (x + 1)(x + 4) + 3$, so the integrand is $x + 4 + 3/(x + 1)$. Integrate: $\mathbf{x^2/2 + 4x + 3 \ln|x + 1| + C}$.

Problem 16 Completing the square

Complete the square: $x^2 + 6x + 13 = (x + 3)^2 + 4$. The integral of $1/(u^2 + 4)$ is $(1/2)\arctan(u/2)$, so the answer is $\mathbf{(1/2) \arctan((x + 3)/2) + C}$.

Problem 17 Logarithm substitution

Let $u = \ln x$, $du = dx/x$. When $x = 1$, $u = 0$, and when $x = e^2$, $u = 2$. So the integral is $\int_0^2 u \, du = 2^2/2 = \mathbf{2}$.

Problem 18 Distance from a speed table

With three subintervals of width 4, the midpoints are 2, 6 and 10. The estimate is $4[v(2) + v(6) + v(10)] = 4(18 + 40 + 45) = 4(103) = \mathbf{412 \text{ miles}}$.

Problem 19 Displacement and distance

An antiderivative is $s(t) = t^3 - 6t^2 + 9t$. Displacement $= s(4) - s(0) = 4 - 0 = \mathbf{4}$. Since $v = 3(t - 1)(t - 3)$, the particle turns at $t = 1$ and 3 . Positions: $s(1) = 4$, $s(3) = 0$, $s(4) = 4$. Distance $= 4 + 4 + 4 = \mathbf{12}$.

Problem 20 Analyzing an accumulation function

By FTC Part 1, $g' = f$, which is zero at $x = 1$ and $x = 3$. The second derivative is $g'' = f'(x) = 2x - 4$. At $x = 1$: $-2 < 0$, a **local maximum**. At $x = 3$: $2 > 0$, a **local minimum**. Since $g(x) = x^3/3 - 2x^2 + 3x$, $g(1) = \mathbf{4/3}$ and $g(3) = \mathbf{0}$.