

Limits and Continuity: Approaching a Value

Grade 12 – AP Calculus AB math · Worksheet Answer Key · Standards: AP Calculus AB Unit 1

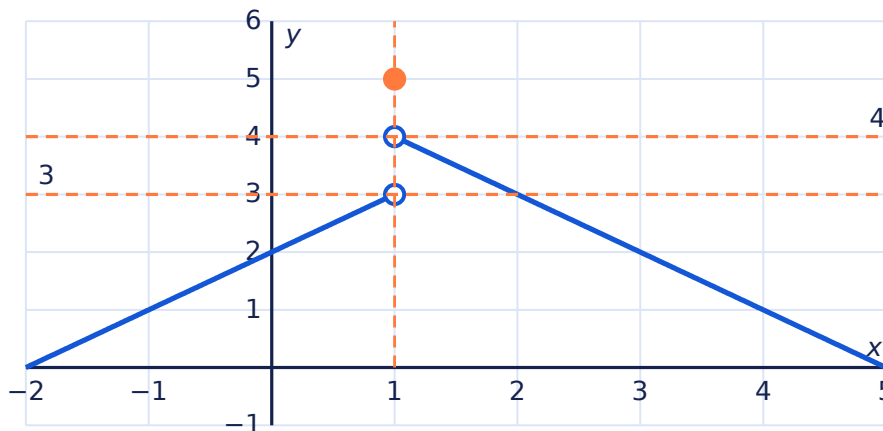
Each solution below shows every step and ends with the final result in bold. If your answer differs, compare the reasoning line by line: most errors come from forgetting to check both sides of a point or from canceling a factor too early.

Standards: AP Calculus AB Unit 1 · **Level:** Grade 12 AP Calculus AB

Problem 1 Limits from a graph

Read the open circles. Approaching from the left the branch heads to height 3; from the right it heads to 4.

- 3**
- 4**
- The one-sided limits differ, so the limit **does not exist**.
- The filled point is at height 5, so $h(1) = 5$.
- Near $x = 3$ the branch is $5 - x$, which approaches **2**.



Dashed lines mark the one-sided limits 3 and 4.

Problem 2 Estimating from a table

As x gets closer to 2 from either side the outputs approach 12 (11.4100 at 1.9, 11.9401 at 1.99, 11.9940 at 1.999, and 12.0060 at 2.001). The estimate is **12**.

Problem 3 Direct substitution

The function is a polynomial, so substitute: $3(4) + 8 + 1 = 12 + 8 + 1 = \mathbf{21}$.

Problem 4 Factoring a 0/0 form

Substitution gives 0/0. Factor the numerator: $(x - 4)(x + 3)$. Cancel $x - 4$ to get $x + 3$, then substitute 4: **7**.

Problem 5 Using a conjugate

Multiply top and bottom by $\sqrt{(x + 25) + 5}$. The numerator becomes $(x + 25) - 25 = x$, so the fraction is $\frac{x}{x(\sqrt{(x + 25) + 5})} = \frac{1}{\sqrt{(x + 25) + 5}}$. At $x = 0$ this is $1/10$: **1/10**.

Problem 6 One-sided limits with absolute value

For $x > 5$, $|2x - 10| = 2(x - 5)$, so $g = 2$. For $x < 5$, $|2x - 10| = -2(x - 5)$, so $g = -2$. The limits are **-2** from the left and **2** from the right, so the two-sided limit **does not exist**.

Problem 7 Limit at infinity of a rational function

Divide every term by x^2 : $\frac{(6 - 1/x)}{(2 + 5/x^2)}$. As $x \rightarrow \infty$, the small terms go to 0, leaving $6/2$. The limit is **3**, so the horizontal asymptote is $y = 3$.

Problem 8 Limits at infinity with a root

For large $|x|$, $\sqrt{x^2 + 9}$ behaves like $|x|$. As $x \rightarrow \infty$, $|x| = x$, so the ratio tends to $4/1 = 4$. As $x \rightarrow -\infty$, $|x| = -x$, so the ratio tends to $4/(-1) = -4$. The limits are **4 and -4**.

Problem 9 Infinite one-sided limits

Neither factor of the denominator cancels with the numerator, so the asymptotes are **$x = 7$ and $x = -3$** . Near 7 the numerator is about 8 and $x + 3$ is about 10, both positive. For $x > 7$ the factor $x - 7$ is a small positive number, so $\lim_{x \rightarrow 7^+} f(x) = +\infty$. For $x < 7$ it is a small negative number, so $\lim_{x \rightarrow 7^-} f(x) = -\infty$.

Problem 10 Squeeze Theorem

Since $-1 \leq \cos(7/x) \leq 1$, multiplying by x^2 (which is nonnegative) gives $-x^2 \leq x^2 \cos(7/x) \leq x^2$. Both bounds approach 0 as $x \rightarrow 0$, so the limit is **0**.

Problem 11 Making a function continuous

Factor: $x^2 - 2x - 35 = (x - 7)(x + 5)$, so $f(x) = x + 5$ for $x \neq 7$ and the limit at 7 is 12. Continuity requires $f(7) = \text{limit}$, so $k = \mathbf{12}$.

Problem 12 Classifying discontinuities

The function is undefined at -5 and at 2 . At $x = -5$ the factor cancels, leaving $\frac{(x - 6)}{(x - 2)}$, whose value at -5 is $-11/-7 = 11/7$. That is a **removable discontinuity** with limit $11/7$. At $x = 2$ the factor $x - 2$ remains and the numerator tends to -4 , so there is an **infinite discontinuity** (vertical asymptote): $-\infty$ from the right, $+\infty$ from the left.

Problem 13 A piecewise function and one parameter

The right limit is $7(2) - 9 = 5$. The left limit equals $f(2) = 4 + 2a$. Set $4 + 2a = 5$, so $a = \mathbf{1/2}$.

Problem 14 Intermediate Value Theorem for a polynomial

f is a polynomial, so it is continuous on $[1, 2]$. $f(1) = 1 + 3 - 11 = -7$ and $f(2) = 8 + 6 - 11 = 3$. Since 0 is between -7 and 3 , the Intermediate Value Theorem guarantees a number c in $(1, 2)$ with $f(c) = 0$. **A zero exists in $(1, 2)$.**

Problem 15 Counting guaranteed zeros

(a) The sign changes between 0 and 2 (-3 to 4), between 2 and 4 (4 to -1), and between 4 and 6 (-1 to 5). Each interval has at least one zero, so at least **3 zeros**.

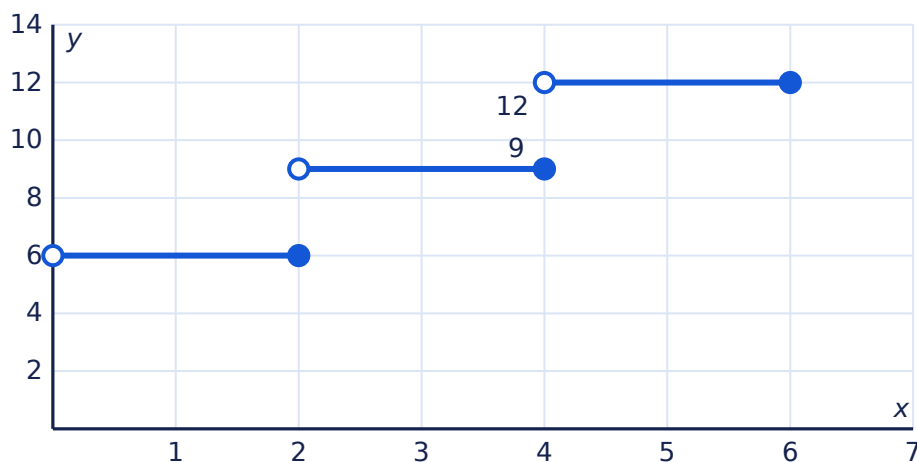
(b) On $[4, 6]$, 4.5 lies between -1 and 5 , so the theorem **guarantees a solution**. On $[0, 2]$, 4.5 is not between -3 and 4 , so **no solution is guaranteed**.

Problem 16 Parking garage fees

(a) The cost just below 4 hours is $\$9$ and just above is $\$12$, so the limits are **9** and **12**.

(b) At $t = 3$ the cost is constant at 9 nearby, so C is **continuous at 3**. At $t = 4$ the one-sided limits differ, so C has a **jump discontinuity at 4**.

(c) 4.5 is in $(4, 6]$, so the cost is **\\$12**.



The jump at $t = 4$ goes from \$9 to \$12.

Problem 17 Medicine level in the blood

$C(0) = 20/10 = 2$ mg/L. As $t \rightarrow \infty$ the ratio of leading coefficients is $45/3 = 15$, so the long-run level is **15 mg/L**. Solve $45t + 20 = 14(3t + 10) = 42t + 140$, so $3t = 120$ and $t = 40$ hours. The level approaches 15 but never reaches it.

Problem 18 Repairing a cubic quotient

Factor: $\frac{(x-3)(x^2+3x+9)}{(x-3)(x+3)}$. At $x = 3$ the factor cancels, and the limit is $\frac{(9+9+9)}{6} = 9/2$ — a **removable discontinuity**; define $f(3) = 9/2$. At $x = -3$ the numerator tends to 9 while the denominator tends to 0, so it is an **infinite discontinuity** with a vertical asymptote: the limit is $-\infty$ from the left and $+\infty$ from the right.

Problem 19 Horizontal and vertical asymptotes with a root

For large $|x|$, $\sqrt{9x^2 + 4x} \approx 3|x|$. As $x \rightarrow \infty$ the ratio tends to **3**; as $x \rightarrow -\infty$, $|x| = -x$, so it tends to **-3**. The horizontal asymptotes are $y = 3$ and $y = -3$. At $x = -1$ the radicand is $9 - 4 = 5$, so the numerator is $\sqrt{5} > 0$ while $x + 1 \rightarrow 0$. Hence $\lim_{x \rightarrow -1^+} f(x) = +\infty$ and $\lim_{x \rightarrow -1^-} f(x) = -\infty$: a **vertical asymptote at $x = -1$** .

Problem 20 Continuity of a three-piece function

At $x = 2$ the right limit is $4 + 1 = 5$ and the middle piece gives $2b + 3$. So $2b + 3 = 5$ and $b = 1$. At $x = -1$ the middle piece gives $-b + 3 = 2$, and the left piece gives $-2 + a$. So $-2 + a = 2$ and $a = 4$. The values are **$a = 4$ and $b = 1$** .