

Motion, Related Rates and Linearization

Grade 12 – AP Calculus AB math · Lesson · Standards: AP Calculus AB Unit 4

Quick answer: The derivative gives the rate of change of a quantity with units of output per input. In context it describes velocity and acceleration, drives related rates problems, builds the linear approximation near a point, and powers L'Hôpital's Rule for $0/0$ and ∞/∞ limits in AP Calculus AB Unit 4.

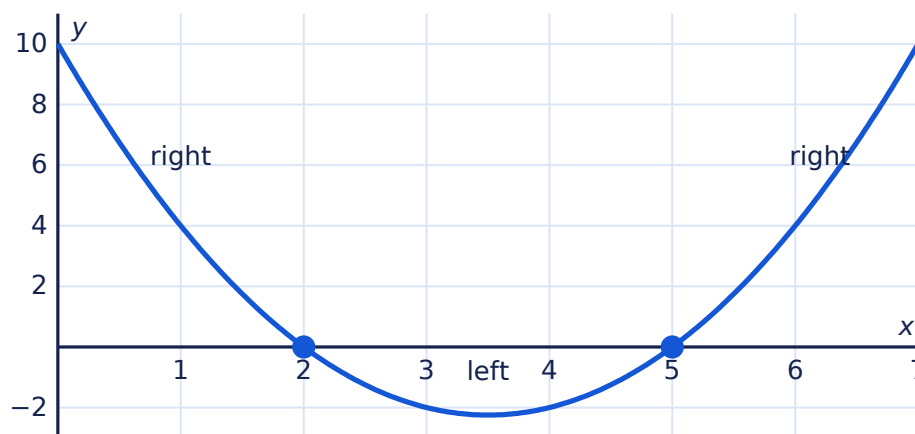
Standards: AP Calculus AB Unit 4 · **Level:** Grade 12 AP Calculus AB

What does a derivative mean in context?

The derivative $f'(t)$ is the instantaneous rate of change of f with units of output per input. If $C(q)$ is the cost in dollars of making q pens and $C'(400) = 0.35$, then the 401st pen costs about \$0.35 more.

Motion vocabulary

For a position function $s(t)$: velocity $v = s'$, acceleration $a = v' = s''$, and speed $= |v|$. The particle speeds up when v and a have the same sign and slows down when the signs differ.



With $v(t) = t^2 - 7t + 10 = (t - 2)(t - 5)$, the particle moves right, then left, then right again.

How do related rates work?

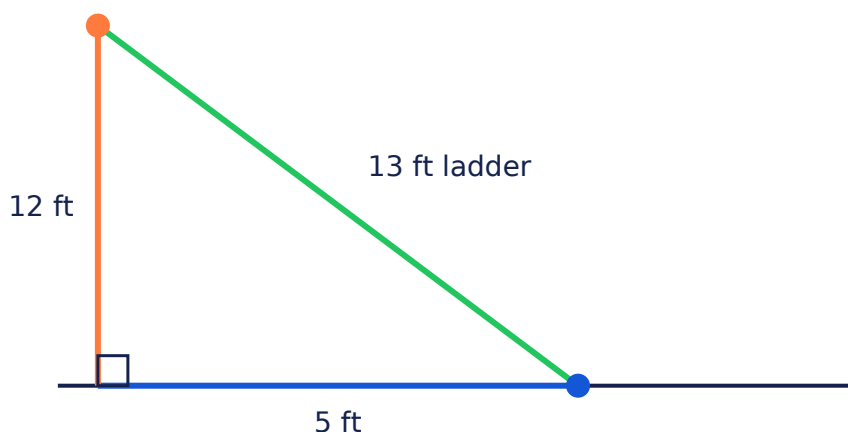
Method

1. Draw a picture and name each changing quantity with a variable.
2. Write one equation that relates the variables, using a formula or similar triangles.

3. Differentiate both sides with respect to time, then substitute the known values.

Worked example

A 13-foot ladder rests on a wall. Its base is 5 feet away and slides outward at 2 ft/s. From $x^2 + y^2 = 169$, differentiate: $2x \frac{dx}{dt} + 2y \frac{dy}{dt} = 0$. With $x = 5$ and $y = 12$: $20 + 24 \frac{dy}{dt} = 0$, so the top slides down at **5/6 ft/s**.

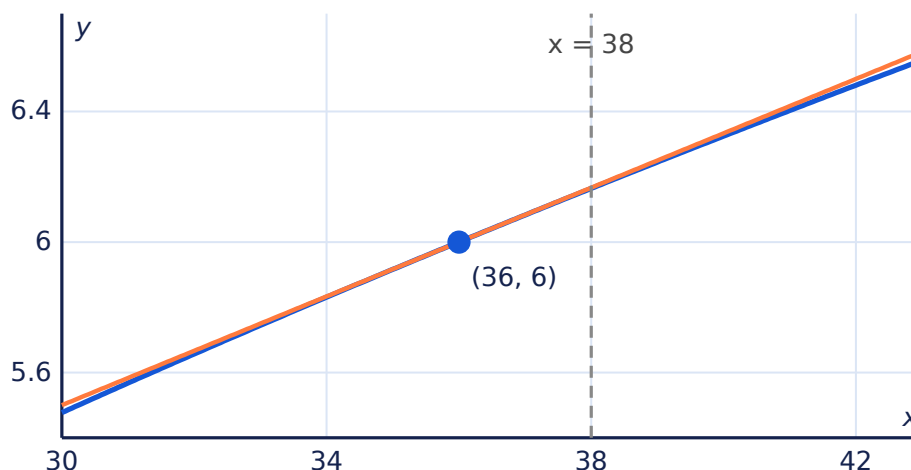


The ladder triangle at the moment the base is 5 feet from the wall.

What are local linearization and L'Hôpital's Rule?

Linearization and L'Hôpital

Near $x = a$, $f(x) \approx f(a) + f'(a)(x - a)$. If a limit of a quotient has the form $0/0$ or ∞/∞ , then $\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$, when the second limit exists.



The tangent line at (36, 6) estimates $\sqrt{38} \approx 6 + 2/12 = 6.1667$. The curve bends downward, so the estimate is slightly high.

For example, $\lim_{x \rightarrow 1} \frac{(\ln x)}{(x - 1)}$ is $0/0$, and the derivative quotient $\frac{1/x}{1}$ tends to **1**.

Common mistake

L'Hôpital's Rule differentiates the numerator and denominator separately. It is not the quotient rule, and it is only allowed for indeterminate forms.

Mathbot's tip

Always write the units next to a derivative. "Dollars per pen" or "feet per second" tells you what the number means.

Key points to remember

- Velocity is the derivative of position, and acceleration is the derivative of velocity.
- A particle speeds up when velocity and acceleration have the same sign.
- Related rates problems differentiate one equation with respect to time.
- Local linearization uses $f(a) + f'(a)(x - a)$ to estimate nearby values.
- L'Hôpital's Rule applies only to $0/0$ or ∞/∞ forms.

Motion, Related Rates and Linearization: frequently asked questions

How do I interpret a derivative in a sentence?

State the quantity, the input value, the units and the direction of change. For example, $C'(400) = 0.35$ means the cost is increasing at about \$0.35 per pen when 400 pens are made.

How can I tell whether a particle is speeding up?

Compare the signs of velocity and acceleration at that time. Same signs mean speeding up, opposite signs mean slowing down. A zero in either one marks a change.

When do I substitute numbers in a related rates problem?

Substitute only after differentiating. Plugging in a value that is changing before you differentiate turns a variable into a constant and destroys its rate.

Is a linear approximation an overestimate or an underestimate?

If the graph is concave up, the tangent line lies below the curve and gives an underestimate. If it is concave down, the tangent line lies above and gives an overestimate.

When can I use L'Hôpital's Rule?

Use it when direct substitution gives $0/0$ or ∞/∞ . Check the form first, then differentiate the top and bottom separately and take the new limit.