

Motion, Related Rates and Linearization

Grade 12 – AP Calculus AB math · Worksheet Answer Key · Standards: AP Calculus AB Unit 4

Each solution states the equation used, the differentiation step and the units. If your answer differs, check whether values were substituted before differentiating, which is the most common error in related rates.

Standards: AP Calculus AB Unit 4 · **Level:** Grade 12 AP Calculus AB

Problem 1 Interpreting marginal cost

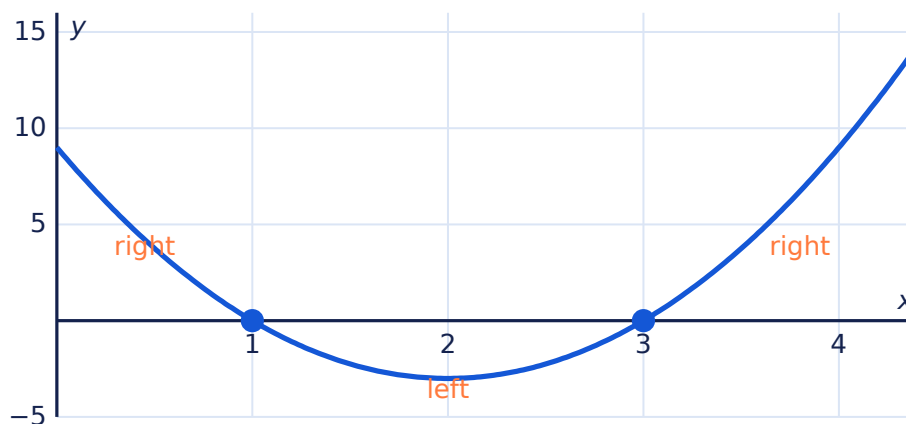
When 200 cupcakes are being baked, the cost is increasing at a rate of **\$1.85 per cupcake**. That is, baking the 201st cupcake adds about \$1.85 to the total cost.

Problem 2 Velocity and acceleration at an instant

$v(t) = 3t^2 - 18t + 15$, so $v(2) = 12 - 36 + 15 = -9$ ft/s. $a(t) = 6t - 18$, so $a(2) = -6$ ft/s². The speed is $|-9| = 9$ ft/s. Both are negative, so the cart is speeding up.

Problem 3 Direction of motion

$v(t) = 3t^2 - 12t + 9 = 3(t - 1)(t - 3)$, which is zero at $t = 1$ and $t = 3$. Testing a point in each interval gives $v(0) = 9 > 0$, $v(2) = -3 < 0$ and $v(4) = 9 > 0$. The particle moves **right on $[0, 1)$ and $(3, \infty)$ and left on $(1, 3)$** .



Direction labels from the sign of v .

Problem 4 Acceleration from a table

The nearest times around 3.5 are 2 and 5: $\frac{(16 - 7)}{(5 - 2)} = 9/3 = \mathbf{3 \text{ m/s}^2}$.

Problem 5 Speeding up or slowing down

$a(t) = 2t - 8$. At $t = 1$: $v = 5$ and $a = -6$, opposite signs, so the particle is **slowing down**. At $t = 7$: $v = 5$ and $a = 6$, same sign, so it is **speeding up**.

Problem 6 Linearization of a cube root

$f(27) = 3$ and $f'(x) = 1/(3x^{2/3})$, so $f'(27) = 1/27$. The linearization is $L(x) = 3 + (x - 27)/27$. Then $\sqrt[3]{29} \approx 3 + 2/27 \approx \mathbf{3.0741}$. The cube root graph is concave down, so the tangent line lies above it and the estimate is **too high**.

Problem 7 Linear approximation of an exponential

$g(0) = 1$ and $g'(x) = 5e^{5x}$, so $g'(0) = 5$. Then $L(x) = 1 + 5x$ and $g(0.02) \approx \mathbf{1.1}$. Since $g'' = 25e^{5x} > 0$, the graph is concave up and the tangent line lies below it: an **underestimate** (actual ≈ 1.1052).

Problem 8 L'Hôpital's Rule once

Direct substitution gives $0/0$. Differentiate top and bottom: $\frac{4e^{4x}}{(6 \cos 6x)}$. At $x = 0$ this is $4/6$. The limit is $\mathbf{2/3}$.

Problem 9 L'Hôpital's Rule twice

The form is ∞/∞ . First pass: $\frac{2x}{3e^{3x}}$, still ∞/∞ . Second pass: $\frac{2}{9e^{3x}}$, which tends to 0. The limit is $\mathbf{0}$.

Problem 10 Growing circle

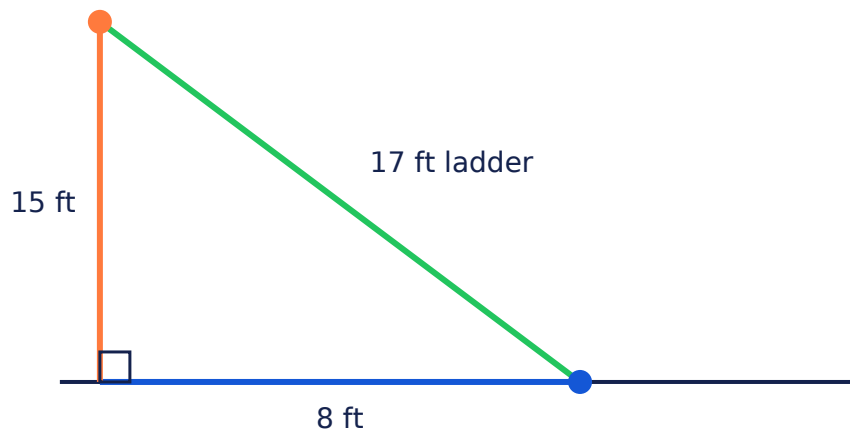
$A = \pi r^2$, so $dA/dt = 2\pi r \, dr/dt = 2\pi(15)(0.4) = \mathbf{12\pi \text{ cm}^2/\text{s}}$ (about $37.7 \text{ cm}^2/\text{s}$).

Problem 11 Inflating a balloon

$dV/dt = 4\pi r^2 \, dr/dt$. Substitute: $90 = 4\pi(9) \, dr/dt = 36\pi \, dr/dt$. So $dr/dt = 90/(36\pi) = \mathbf{5/(2\pi) \text{ in/s}}$ $\approx 0.80 \text{ in/s}$.

Problem 12 Sliding ladder

$x^2 + y^2 = 289$. When $x = 8$, $y = \sqrt{289 - 64} = 15$. Differentiate: $x \frac{dx}{dt} + y \frac{dy}{dt} = 0$, so $8(1.5) + 15 \frac{dy}{dt} = 0$ and $\frac{dy}{dt} = -12/15 = -\mathbf{0.8 \text{ ft/s}}$. The top slides down at 0.8 ft/s.



The height is 15 feet at that moment.

Problem 13 Marginal cost and actual cost

$C'(x) = 0.04x + 4$, so $C'(100) = \mathbf{\$8 \text{ per skateboard}}$. It estimates the extra cost of the 101st skateboard. The actual cost is $C(101) - C(100) = 8.02$, so the estimate is within $\mathbf{\$0.02}$.

Problem 14 Filling a conical tank

By similar triangles $r = h/2$, so $V = (1/3)\pi(h/2)^2h = \pi h^3/12$. Differentiate: $dV/dt = (\pi h^2/4) dh/dt$. Substitute: $20 = (16\pi/4) dh/dt = 4\pi dh/dt$, so $dh/dt = \mathbf{5/\pi \text{ ft/min}} \approx 1.59 \text{ ft/min}$.

Problem 15 Relative growth of a population

$P(t) = 320e^{0.04t}$. At $t = 5$, $P'(5) = 320e^{0.2} \approx \mathbf{390.85 \text{ people per year}}$. The relative rate is $320e^{0.04t}/(8,000e^{0.04t}) = \mathbf{0.04}$, so the population grows 4% per year at all times.

Problem 16 Shadow of a person walking

Let T be the distance from the post to the shadow tip. Similar triangles give $\frac{(T-x)}{6} = \frac{T}{15}$. So $15T - 15x = 6T$, $9T = 15x$, and $T = (5/3)x$. Then $dT/dt = (5/3)(4) = \mathbf{20/3 \text{ ft/s}}$. The shadow's length is $T - x$, which grows at $20/3 - 4 = \mathbf{8/3 \text{ ft/s}}$.

Problem 17 Distance traveled by a particle

$v(t) = 3(t-2)(t-6)$: the particle moves right on $[0, 2]$, left on $[2, 6]$, and right on $[6, 7]$.

Positions: $s(0) = 0$, $s(2) = 32$, $s(6) = 0$, $s(7) = 7$. Distances: $32 + 32 + 7 = \mathbf{71}$. The net displacement is $7 - 0 = \mathbf{7}$.

Problem 18 Trigonometric limit with two passes

The first form is $0/0$. Differentiate: $(5 \sin 5x)/_{2x}$, still $0/0$. Differentiate again: $(25 \cos 5x)/_2$. At $x = 0$ this is $25/2$. The limit is $\mathbf{25/2}$.

Problem 19 Intervals of speeding up

$v(t) = 6(t - 2)(t - 3)$ and $a(t) = 12t - 30$, which is zero at $t = 2.5$. Signs: on $(0, 2)$, $v > 0$, $a < 0$ (slowing); on $(2, 2.5)$, $v < 0$, $a < 0$ (speeding up); on $(2.5, 3)$, $v < 0$, $a > 0$ (slowing); on $(3, 4)$, $v > 0$, $a > 0$ (speeding up). The particle speeds up on **(2, 2.5) and (3, 4)**.

Problem 20 Distance from a radar station

Let x be the ground distance and D the straight-line distance. Then $D^2 = x^2 + 16$, so $2D \, dD/dt = 2x \, dx/dt$. When $x = 3$, $D = 5$. Therefore $dD/dt = (3)(500)/5 = \mathbf{300 \text{ mph}}$.