

Solving Any Triangle with Sines, Cosines and Area

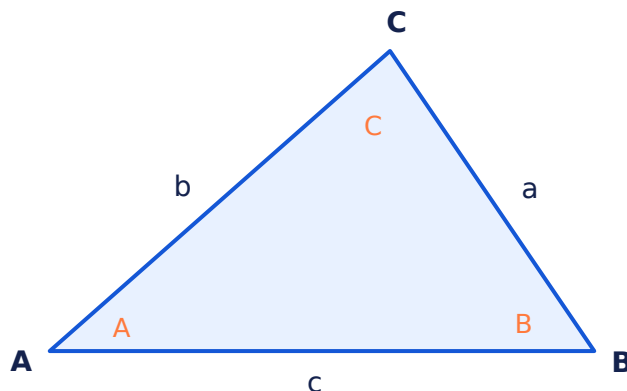
Grade 10 – Geometry math · Lesson · Standards: G-SRT.D.9, G-SRT.D.10, G-SRT.D.11

Quick answer: The Law of Sines and Law of Cosines extend trigonometry to triangles without a right angle. Use sines when you know an angle and its opposite side, use cosines for SAS or SSS, and find the area with one half times two sides times the sine of the included angle.

Standards: G-SRT.D.9, G-SRT.D.10, G-SRT.D.11 · **Level:** Grade 10 Geometry

How are the sides and angles of any triangle labeled?

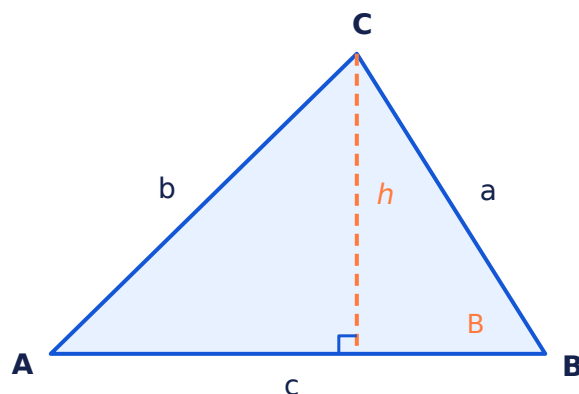
When a triangle has no right angle, SOH CAH TOA does not apply directly. Instead, name each side with the lowercase letter of the angle it faces.



Side a faces angle A, side b faces B, side c faces C.

How do you find the area of a triangle from two sides and the angle between them?

Drop an altitude h from C to side c. In the small right triangle, $h = a \sin B$, and the area is half the base times the height.



The altitude h equals $a \sin B$.

Area formula

$$\text{Area} = \frac{1}{2} a b \sin C$$

Use the two sides that form the angle.

What do the Law of Sines and Law of Cosines say?

Law of Sines

$$\frac{(\sin A)}{a} = \frac{(\sin B)}{b} = \frac{(\sin C)}{c}$$

Law of Cosines

$$c^2 = a^2 + b^2 - 2ab \cos C$$

When $C = 90^\circ$, $\cos C = 0$ and this becomes the Pythagorean Theorem.

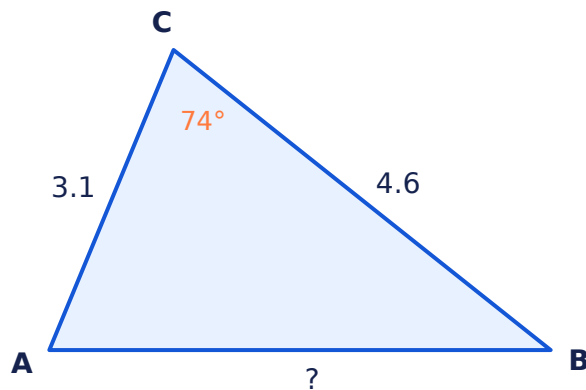
Given	Start with
Two angles and any side (AAS, ASA)	Law of Sines
Two sides and the angle between them (SAS)	Law of Cosines
Three sides (SSS)	Law of Cosines
Two sides and an angle not between them (SSA)	Law of Sines, but there may be two triangles

Example

Two trailheads sit 4.6 miles and 3.1 miles from a lake lodge, and the lines from the lodge to them meet at 74° . How far apart are the trailheads?

1. The known angle lies between the known sides, so use the Law of Cosines.
2. $c^2 = 4.6^2 + 3.1^2 - 2(4.6)(3.1) \cos 74^\circ$.
3. $c^2 \approx 21.16 + 9.61 - 28.52 \cdot 0.2756 \approx 22.91$.

4. $c \approx 4.8$ miles.



Lodge angle 74°, unknown distance between trailheads.

Common mistake

Subtracting before multiplying. In the Law of Cosines, finish $2ab \cos C$ first, then subtract it from $a^2 + b^2$.

Mathbot's tip Find the largest side or angle last. Angles from the Law of Cosines are never ambiguous, but sine can hide an obtuse angle.

Key points to remember

- Area = $\frac{1}{2}ab \sin C$, where C is the angle between sides a and b.
- Law of Sines: $\sin A / a = \sin B / b = \sin C / c$.
- Law of Cosines: $c^2 = a^2 + b^2 - 2ab \cos C$.
- SSA can produce two different triangles, so check for an obtuse angle.

Solving Any Triangle with Sines, Cosines and Area: frequently asked questions

When do I use the Law of Cosines instead of the Law of Sines?

Use the Law of Cosines when you know two sides and the angle between them or when you know all three sides. The Law of Sines needs a known side and its opposite angle as a pair.

Why does the area formula contain $\sin C$?

The altitude from the opposite vertex equals one side times the sine of the included angle. Half of base times that height gives $\frac{1}{2}ab \sin C$.

Can the Law of Cosines find an obtuse angle?

Yes. When the cosine comes out negative, the inverse cosine returns an angle between 90° and 180° automatically, so there is no ambiguity.

What is the ambiguous case?

If you know two sides and an angle opposite one of them, the data may fit two triangles, one, or none. The inverse sine gives an acute angle, and its supplement may also work.

How does the Law of Cosines relate to the Pythagorean Theorem?

When the included angle is 90° , $\cos C = 0$ and the middle term disappears. What remains is $c^2 = a^2 + b^2$.