

Complex Factoring and Sphere Volume

Grade 12 – Precalculus math · Lesson · Standards: N-CN.C.8, G-GMD.A.2

Quick answer: Because $i^2 = -1$, a sum of squares factors over the complex numbers: $a^2 + b^2 = (a + bi)(a - bi)$. So $x^2 + 4 = (x + 2i)(x - 2i)$. The volume of a sphere, $(4/3)\pi r^3$, follows from Cavalieri's principle by comparing a hemisphere with a cylinder minus a cone.

Standards: N-CN.C.8, G-GMD.A.2 · **Level:** Grade 12 Precalculus

How do the algebra identities extend to complex numbers?

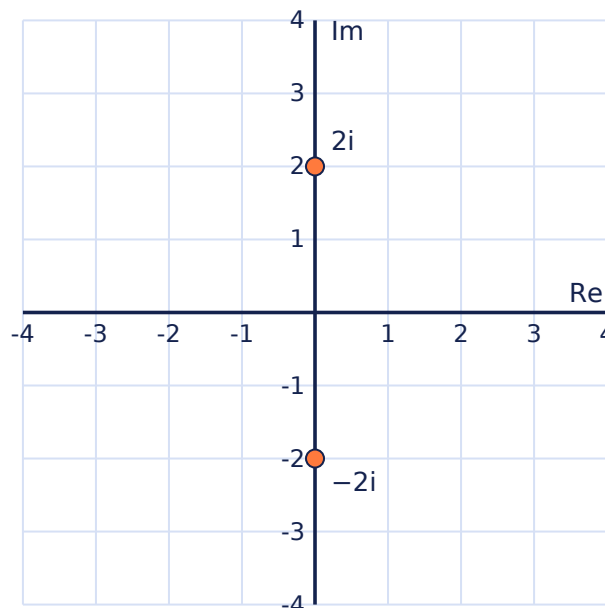
Since $i^2 = -1$, a sum of squares is no longer prime. The difference-of-squares pattern $a^2 - b^2 = (a + b)(a - b)$ becomes $a^2 + b^2 = (a + bi)(a - bi)$. Check: $(a + bi)(a - bi) = a^2 - b^2 i^2 = a^2 + b^2$.

Sum of squares over the complex numbers

$a^2 + b^2 = (a + bi)(a - bi)$. In particular, $x^2 + k = (x + \sqrt{k}i)(x - \sqrt{k}i)$ for any $k > 0$.

Worked example

$x^2 + 4 = (x + 2i)(x - 2i)$. Setting each factor to zero gives the solutions $x = \pm 2i$, which sit on the imaginary axis of the complex plane.



The zeros of $x^2 + 4$ are $2i$ and $-2i$.

How do you factor polynomials completely over the complex numbers?

Method

1. Factor out any common factor and apply real factoring first, such as a difference of squares.
2. Split every sum of squares with $a^2 + b^2 = (a + bi)(a - bi)$.
3. For a quadratic with a negative discriminant, use the quadratic formula; the zeros come in conjugate pairs.
4. Multiply the factors back to check.

For example, $x^4 - 16 = (x^2 - 4)(x^2 + 4) = (x - 2)(x + 2)(x + 2i)(x - 2i)$, so it has four zeros: ± 2 and $\pm 2i$.

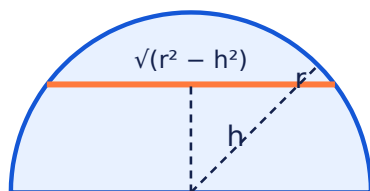
Common mistake

Writing $x^2 + 9 = (x + 3)^2$. Expanding gives $x^2 + 6x + 9$, which has a middle term. The correct factors have opposite signs: $(x + 3i)(x - 3i)$.

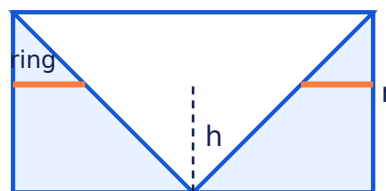
Why is the volume of a sphere $(4/3)\pi r^3$?

Cavalieri's principle says two solids with the same height have equal volumes if every horizontal slice has equal area. Compare a hemisphere of radius r with a cylinder of radius r and height r that has an upside-down cone removed. At height h , the hemisphere's slice is a disk of radius $\sqrt{r^2 - h^2}$, with area $\pi(r^2 - h^2)$. The cylinder-minus-cone slice is a ring with outer radius r and inner radius h , with area $\pi r^2 - \pi h^2 = \pi(r^2 - h^2)$. The areas match at every height.

Hemisphere, radius r

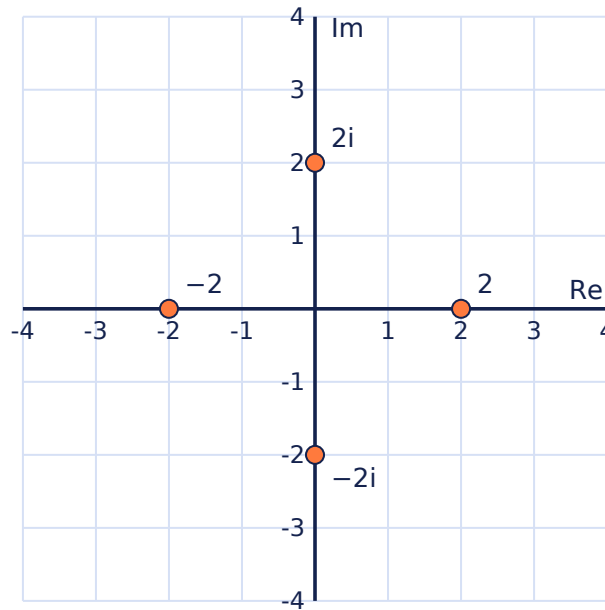


Cylinder minus cone



At every height h both slices have the same area, $\pi(r^2 - h^2)$.

So the hemisphere's volume equals the cylinder minus the cone: $\pi r^3 - \frac{1}{3}\pi r^3 = \frac{2}{3}\pi r^3$. A full sphere doubles this: $V = \frac{4}{3}\pi r^3$.



The four zeros of $x^4 - 16$ form a square around the origin.

Mathbot's tip

When a sum of squares appears, look for its square roots, then give the factors opposite signs and add an i to the second root.

Key points to remember

- $i^2 = -1$, and powers of i repeat in a cycle of 4: $i, -1, -i, 1$.
- $a^2 + b^2 = (a + bi)(a - bi)$; for example $x^2 + 25 = (x + 5i)(x - 5i)$.
- Complete factoring over the complex numbers splits every sum of squares, so degree n gives n zeros.
- Cavalieri's principle: equal heights and equal slice areas at every level mean equal volumes.
- Hemisphere = cylinder – cone = $(2/3)\pi r^3$, so a sphere has volume $(4/3)\pi r^3$.

Complex Factoring and Sphere Volume: frequently asked questions

Why can $x^2 + 4$ be factored over the complex numbers?

Because $(x + 2i)(x - 2i) = x^2 - 4i^2 = x^2 + 4$. The imaginary unit supplies the negative sign that a sum of squares lacks over the real numbers.

How do I evaluate a large power of i ?

The powers repeat every four steps: $i, -1, -i, 1$. Divide the exponent by 4 and use the remainder. For example, i^{27} has remainder 3, so $i^{27} = i^3 = -i$.

What does Cavalieri's principle say?

If two solids have the same height and the horizontal slices at every height have equal areas, the solids have equal volumes. It lets you find a hard volume from an easy one.

Why do the hemisphere and the cylinder-minus-cone have the same slices?

At height h , the hemisphere's slice has radius $\sqrt{r^2 - h^2}$ by the Pythagorean Theorem, giving area $\pi(r^2 - h^2)$. The ring has area $\pi r^2 - \pi h^2$, which is the same expression.

How many zeros does a polynomial have over the complex numbers?

A polynomial of degree n has exactly n zeros when repeated zeros are counted. For $x^4 - 16$ they are 2 , -2 , $2i$ and $-2i$.