

Polar Form and Powers of Complex Numbers

Grade 12 – Precalculus math · Lesson · Standards: N-CN.B.4, N-CN.B.5, N-CN.B.6, N-CN.A.3

Quick answer: A complex number $a + bi$ can be written in polar form $r(\cos \theta + i \sin \theta)$, where r is its distance from the origin and θ is its angle from the positive real axis. In polar form, multiplication multiplies moduli and adds angles, and powers follow De Moivre's theorem.

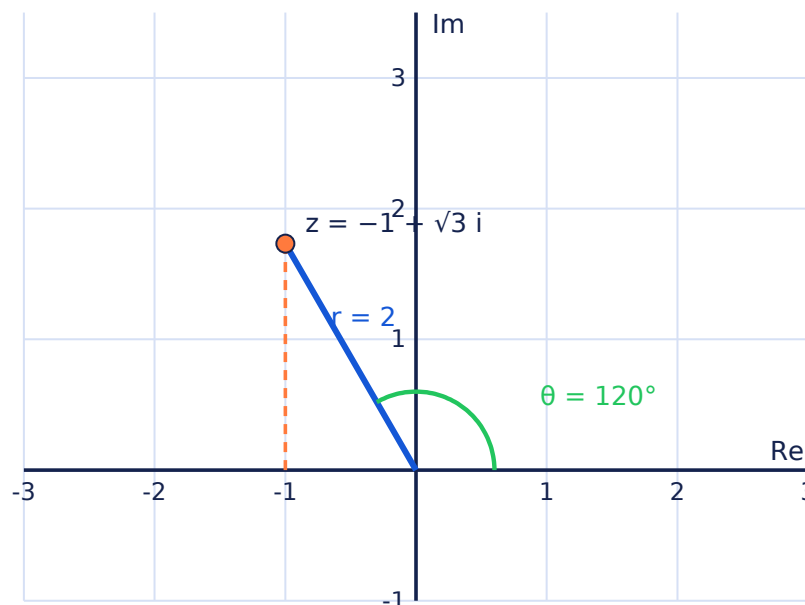
Standards: N-CN.B.4, N-CN.B.5, N-CN.B.6, N-CN.A.3 · **Level:** Grade 12 Precalculus

How do you read a complex number on the complex plane?

The complex plane has a horizontal real axis and a vertical imaginary axis. The number $z = a + bi$ sits at the point (a, b) . Its distance from the origin is the modulus, and the angle from the positive real axis, measured counterclockwise, is the argument.

Modulus, argument, conjugate

$|z| = r = \sqrt{a^2 + b^2}$. The argument θ satisfies $\tan \theta = \frac{b}{a}$ with θ placed in the correct quadrant. The conjugate is $\bar{z} = a - bi$, the mirror image across the real axis, and $z \cdot \bar{z} = |z|^2$.



The point $-1 + \sqrt{3}i$ lies in Quadrant II, 2 units from the origin, at 120° .

How do you switch between rectangular and polar form?

Polar (trigonometric) form

$z = r(\cos \theta + i \sin \theta)$, so $a = r \cos \theta$ and $b = r \sin \theta$.

Method: rectangular to polar

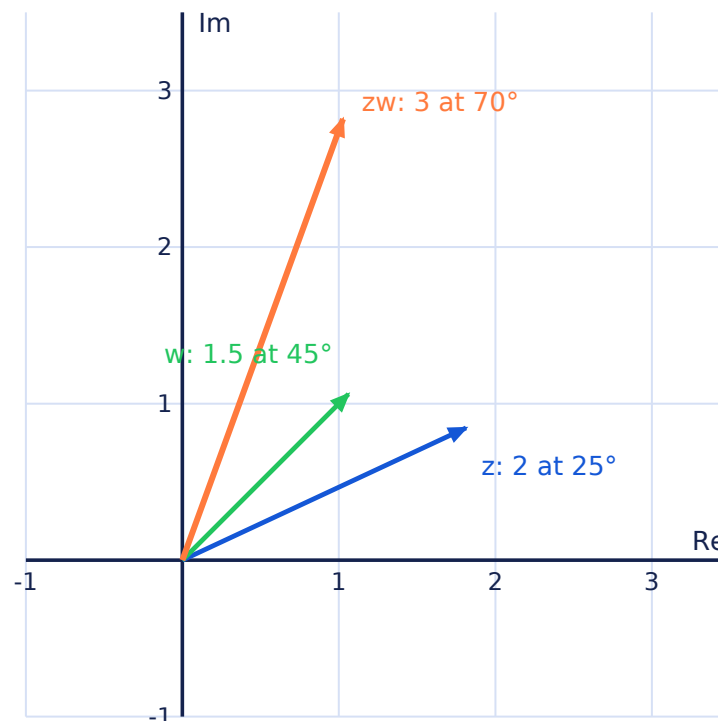
1. Find $r = \sqrt{a^2 + b^2}$.
2. Find the reference angle $\tan^{-1}(|b| \div |a|)$.
3. Use the signs of a and b to pick the quadrant and get θ .
4. Write $r(\cos \theta + i \sin \theta)$.

Worked example

For $-1 + \sqrt{3}i$: $r = \sqrt{1 + 3} = 2$. The reference angle is $\tan^{-1}(\sqrt{3}) = 60^\circ$, and Quadrant II gives $\theta = 180^\circ - 60^\circ = 120^\circ$. So $-1 + \sqrt{3}i = 2(\cos 120^\circ + i \sin 120^\circ)$.

How do you multiply, divide and take powers in polar form?

Operation	Rule for $z_1 = r_1 \angle \theta_1$ and $z_2 = r_2 \angle \theta_2$
Product	$r_1 r_2 \angle (\theta_1 + \theta_2)$
Quotient	$r_1 / r_2 \angle (\theta_1 - \theta_2)$
Power (De Moivre)	$r^n \angle n\theta$
Conjugate	$r \angle (-\theta)$



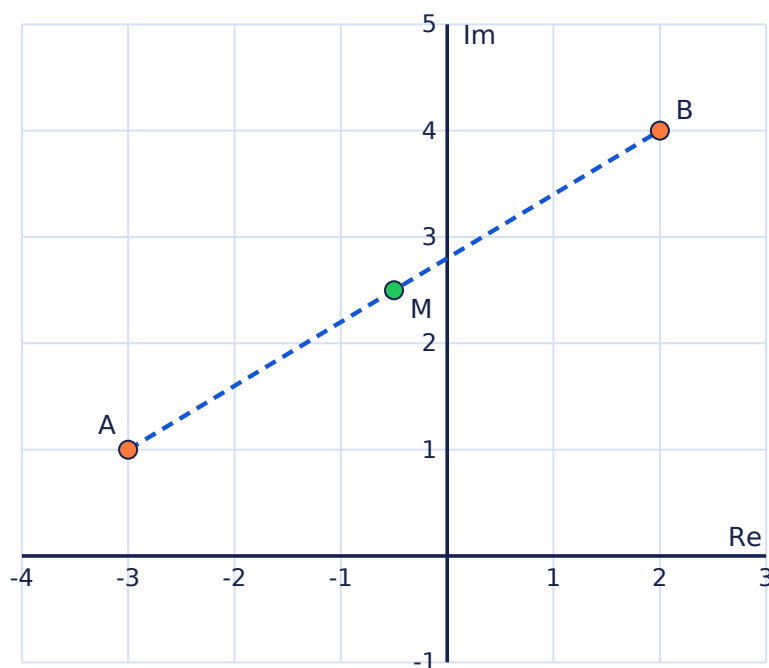
Multiplying multiplies the moduli ($2 \times 1.5 = 3$) and adds the arguments ($25^\circ + 45^\circ = 70^\circ$).

Common mistake

Adding the moduli when multiplying. Moduli multiply; only the angles add.

How do you find distance and midpoint between two complex numbers?

For $z_1 = a + bi$ and $z_2 = c + di$, the distance is $|z_2 - z_1| = \sqrt{(c - a)^2 + (d - b)^2}$, and the midpoint is $\frac{(z_1 + z_2)}{2}$. For $A = -3 + i$ and $B = 2 + 4i$, the distance is $\sqrt{(25 + 9)} = \sqrt{34} \approx 5.83$, and the midpoint is $-\frac{1}{2} + \frac{5}{2}i$.



The distance between A and B is the length of the dashed segment. M is the midpoint.

Mathbot's tip

Think of multiplying by $r\angle\theta$ as a stretch by r followed by a turn of θ . Draw the arrow first, then check your algebra against it.

Key points to remember

- Modulus: $|a + bi| = \sqrt{a^2 + b^2}$; conjugate: $a - bi$.
- Polar form: $z = r(\cos \theta + i \sin \theta)$ with $a = r \cos \theta$ and $b = r \sin \theta$.
- Product: multiply moduli, add arguments. Quotient: divide moduli, subtract arguments.
- De Moivre: $[r(\cos \theta + i \sin \theta)]^n = r^n(\cos n\theta + i \sin n\theta)$.
- Distance between two points in the complex plane is the modulus of their difference.

Polar Form and Powers of Complex Numbers: frequently asked

questions

What is the difference between the modulus and the argument?

The modulus is the length of the arrow from the origin to the number, a non-negative real value. The argument is the angle of that arrow, measured counterclockwise from the positive real axis. Together they locate the number just like coordinates do.

Why do the angles add when two complex numbers are multiplied?

Multiplying $\cos \alpha + i \sin \alpha$ by $\cos \beta + i \sin \beta$ and using the sum formulas for sine and cosine gives $\cos(\alpha + \beta) + i \sin(\alpha + \beta)$. Angles add because each factor rotates the arrow.

Can the argument of a complex number be negative?

Yes. An argument of -60° points to the same direction as 300° . Any two arguments that differ by a multiple of 360° describe the same number, so a final answer should name the range used.

What is De Moivre's theorem used for?

It turns a power of a complex number in polar form into a single step: raise the modulus to the power and multiply the angle by the exponent. It also leads to the n distinct n th roots of any nonzero complex number.

How do I find the argument in Quadrant II or III?

Compute the reference angle with the inverse tangent of $|b| \div |a|$. Then subtract it from 180° in Quadrant II, add it to 180° in Quadrant III, or subtract it from 360° in Quadrant IV.