

Polar Form and Powers of Complex Numbers

Grade 12 – Precalculus math · Worksheet Answer Key · Standards: N-CN.B.4, N-CN.B.5, N-CN.B.6, N-CN.A.3

Each answer shows the modulus and angle work, the conversion back to rectangular form where needed, and a quick independent check. Compare every step, not only the last line, so a wrong angle is caught early.

Standards: N-CN.B.4, N-CN.B.5, N-CN.B.6, N-CN.A.3 · **Level:** Grade 12 Precalculus

Problem 1 Modulus of a Complex Number

$$|z| = \sqrt{(2^2 + (-5)^2)} = \sqrt{(4 + 25)} = \sqrt{29} \approx 5.39.$$

Answer: $\sqrt{29} \approx 5.39$

Problem 2 Conjugate and Product

$$\bar{z} = 3 + 7i. \text{ Then } z \cdot \bar{z} = 3^2 + 7^2 = 9 + 49 = 58, \text{ and } |z|^2 = 9 + 49 = 58, \text{ so they are equal.}$$

Answer: $\bar{z} = 3 + 7i$ and $z \cdot \bar{z} = 58 = |z|^2$

Problem 3 Reading a Point

The point is 6 units left and 1 unit up, so $z = -6 + i$. Its conjugate is $-6 - i$. The modulus is $\sqrt{(36 + 1)} = \sqrt{37}$.

Answer: $z = -6 + i, \bar{z} = -6 - i, |z| = \sqrt{37}$

Problem 4 Modulus and Argument

$r = \sqrt{(25 + 25)} = 5\sqrt{2}$. The reference angle is $\tan^{-1}(5 \div 5) = 45^\circ$, and the number is in Quadrant II, so $\theta = 180^\circ - 45^\circ = 135^\circ$.

Answer: $r = 5\sqrt{2}, \theta = 135^\circ$

Problem 5 Distance and Midpoint

Difference: $(4 - (-3)) + (1 - 5)i = 7 - 4i$. Distance = $\sqrt{(49 + 16)} = \sqrt{65} \approx 8.06$.

$$\text{Midpoint} = \frac{((-3 + 4) + (5 + 1)i)}{2} = \frac{1}{2} + 3i.$$

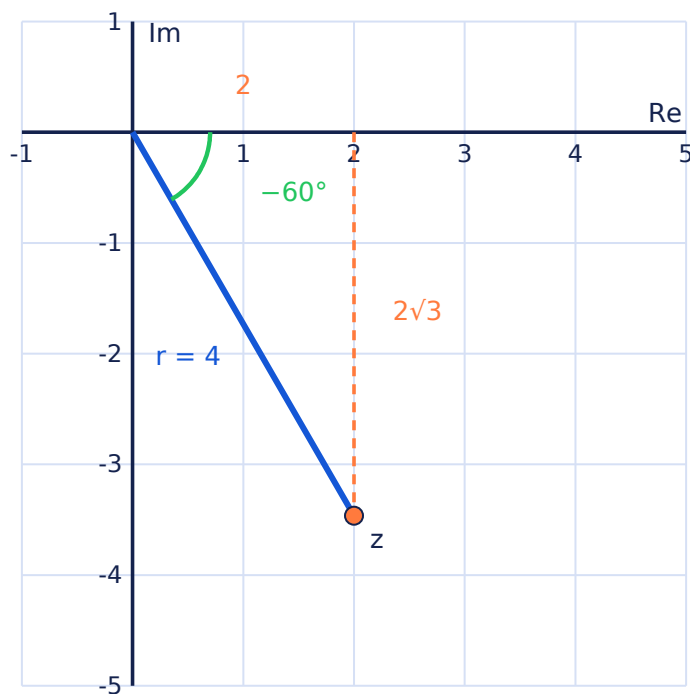
Answer: **$PQ = \sqrt{65}$ and the midpoint is $\frac{1}{2} + 3i$**

Problem 6 Polar to Rectangular

$\cos 60^\circ = \frac{1}{2}$ and $\sin 60^\circ = \frac{\sqrt{3}}{2}$. So $6 \cdot \frac{1}{2} = 3$ and $6 \cdot \frac{\sqrt{3}}{2} = 3\sqrt{3}$.

Answer: **$3 + 3\sqrt{3}i$**

Problem 7 Rectangular to Polar



z is 4 units from the origin at -60° , which is the same direction as 300° .

$r = \sqrt{4 + 12} = 4$. The reference angle is $\tan^{-1}(2\sqrt{3} \div 2) = \tan^{-1}(\sqrt{3}) = 60^\circ$. In Quadrant IV, $\theta = 360^\circ - 60^\circ = 300^\circ$.

Answer: **$4(\cos 300^\circ + i \sin 300^\circ)$**

Problem 8 Multiplying in Polar Form

Multiply moduli: $4 \cdot 5 = 20$. Add angles: $25^\circ + 35^\circ = 60^\circ$. Polar form: $20(\cos 60^\circ + i \sin 60^\circ)$.

Rectangular: $20 \cdot \frac{1}{2} = 10$ and $20 \cdot \frac{\sqrt{3}}{2} = 10\sqrt{3}$.

Answer: **$20(\cos 60^\circ + i \sin 60^\circ) = 10 + 10\sqrt{3}i$**

Problem 9 Dividing in Polar Form

Divide moduli: $18 \div 6 = 3$. Subtract angles: $200^\circ - 80^\circ = 120^\circ$. The quotient is $3(\cos 120^\circ + i \sin 120^\circ)$.

$\cos 120^\circ = -\frac{1}{2}$ and $\sin 120^\circ = \frac{\sqrt{3}}{2}$, so the result is $-\frac{3}{2} + \frac{3\sqrt{3}}{2}i$.

Answer: $-\frac{3}{2} + \frac{3\sqrt{3}}{2}i$

Problem 10 Product via Polar Form

z_1 : $r = \sqrt{2}$, $\theta = 45^\circ$. z_2 : $r = \sqrt{(1+3)} = 2$, $\theta = 60^\circ$.

Product: modulus $\sqrt{2} \cdot 2 = 2\sqrt{2}$, argument $45^\circ + 60^\circ = 105^\circ$.

Check: $z_1 z_2 = (1 - \sqrt{3}) + (1 + \sqrt{3})i$, and its modulus squared is $(1 - \sqrt{3})^2 + (1 + \sqrt{3})^2 = 4 - 2\sqrt{3} + 4 + 2\sqrt{3} = 8$, so the modulus is $2\sqrt{2}$.

Answer: **modulus $2\sqrt{2}$, argument 105°**

Problem 11 A Tenth Power

$1 - i = \sqrt{2}(\cos(-45^\circ) + i \sin(-45^\circ))$. The tenth power has modulus $(\sqrt{2})^{10} = 32$ and angle $10 \cdot (-45^\circ) = -450^\circ$, which is coterminal with -90° .

So the result is $32(\cos(-90^\circ) + i \sin(-90^\circ)) = 32(0 - i) = -32i$.

Check: $(1 - i)^2 = -2i$, and $(-2i)^5 = -32i^5 = -32i$.

Answer: **$-32i$**

Problem 12 Cubing in Polar Form

The modulus becomes $2^3 = 8$ and the angle becomes $3 \cdot 20^\circ = 60^\circ$. So the result is $8(\cos 60^\circ + i \sin 60^\circ) = 8 \cdot \frac{1}{2} + 8 \cdot \frac{\sqrt{3}}{2}i$.

Answer: **$4 + 4\sqrt{3}i$**

Problem 13 A Sixth Power

$r = \sqrt{(3+1)} = 2$ and $\theta = \tan^{-1}(\frac{1}{\sqrt{3}}) = 30^\circ$. Then $(\sqrt{3} + i)^6 = 2^6(\cos 180^\circ + i \sin 180^\circ) = 64(-1)$.

Answer: **-64**

Problem 14 A Fifth Power

$r = \sqrt{2}$ and $\theta = 135^\circ$ (Quadrant II). Then $(-1 + i)^5 = (\sqrt{2})^5(\cos 675^\circ + i \sin 675^\circ)$. Since $(\sqrt{2})^5 = 4\sqrt{2}$ and $675^\circ - 360^\circ = 315^\circ$, this is $4\sqrt{2}(\cos 315^\circ + i \sin 315^\circ) = 4\sqrt{2}(\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i) = 4 - 4i$.

Check: $(-1 + i)^2 = -2i$, $(-1 + i)^4 = -4$, and $-4(-1 + i) = 4 - 4i$.

Answer: **$4 - 4i$**

Problem 15 Cube Roots of 64

Write $64 = 64(\cos 0^\circ + i \sin 0^\circ)$. The cube roots have modulus $\sqrt[3]{64} = 4$ and angles $(0^\circ + 360^\circ k)/3$ for $k = 0, 1, 2$, which are $0^\circ, 120^\circ$ and 240° .

$k = 0$: 4 . $k = 1$: $4(-\frac{1}{2} + \frac{\sqrt{3}}{2}i) = -2 + 2\sqrt{3}i$. $k = 2$: $-2 - 2\sqrt{3}i$.

Answer: **$4, -2 + 2\sqrt{3}i, -2 - 2\sqrt{3}i$**

Problem 16 Trail Distance

$B - A = (5 - (-2)) + (-3 - 6)i = 7 - 9i$. Distance = $\sqrt{(49 + 81)} = \sqrt{130} \approx 11.40$ miles.

Midpoint = $\frac{((-2 + 5) + (6 - 3)i)}{2} = 1.5 + 1.5i$.

Answer: **about 11.40 miles; rest stop at $1.5 + 1.5i$**

Problem 17 Robot Rotation

(a) $2i(3 + 2i) = 6i + 4i^2 = -4 + 6i$.

(b) $(2i)^4 = 16i^4 = 16$, so the position is $16(3 + 2i) = 48 + 32i$.

Check: four 90° turns make a full turn and the distance is multiplied by $2^4 = 16$.

Answer: **(a) $-4 + 6i$; (b) $48 + 32i$**

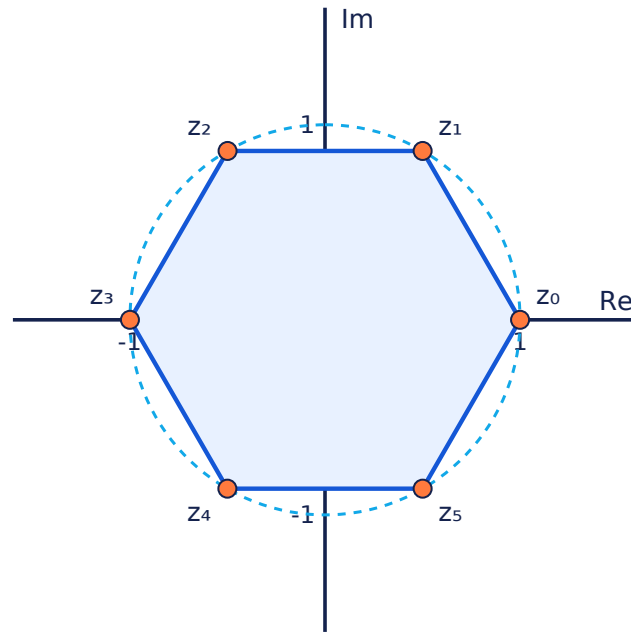
Problem 18 Alternating Current

Divide moduli: $48 \div 6 = 8$. Subtract angles: $70^\circ - 25^\circ = 45^\circ$. In polar form the current is $8(\cos 45^\circ + i \sin 45^\circ)$ amps.

Rectangular: $8 \cdot \frac{\sqrt{2}}{2} = 4\sqrt{2} \approx 5.66$, so the current is about $5.66 + 5.66i$.

Answer: **$8(\cos 45^\circ + i \sin 45^\circ) \approx 5.66 + 5.66i$ amps**

Problem 19 Sixth Roots of Unity



The six sixth roots of 1 form a regular hexagon on the unit circle.

(a) By De Moivre, $z^6 = \cos 360^\circ + i \sin 360^\circ = 1$.

(b) The roots are $\cos \theta + i \sin \theta$ for $\theta = 0^\circ, 60^\circ, 120^\circ, 180^\circ, 240^\circ, 300^\circ$: $1, \frac{1}{2} + \frac{\sqrt{3}}{2}i, -\frac{1}{2} + \frac{\sqrt{3}}{2}i, -1, -\frac{1}{2} - \frac{\sqrt{3}}{2}i, \frac{1}{2} - \frac{\sqrt{3}}{2}i$.

(c) They sit 60° apart on the unit circle, so they form a regular hexagon whose side length equals 1.

Answer: **a regular hexagon with side length 1**

Problem 20 A Quotient of Powers

$1 - \sqrt{3}i = 2(\cos(-60^\circ) + i \sin(-60^\circ))$, so its fifth power is $32(\cos(-300^\circ) + i \sin(-300^\circ)) = 32 \angle 60^\circ$.

$(1 + i)^4 = (\sqrt{2})^4 \angle 180^\circ = 4 \angle 180^\circ = -4$.

Quotient: $8 \angle (60^\circ - 180^\circ) = 8 \angle (-120^\circ) = 8(-\frac{1}{2} - \frac{\sqrt{3}}{2}i) = -4 - 4\sqrt{3}i$.

Answer: **$-4 - 4\sqrt{3}i$**