

# Parabolas, Ellipses and Hyperbolas in Standard Form

Grade 12 – Precalculus math · Lesson · Standards: G-GPE.A.1, G-GPE.A.2, G-GPE.A.3

**Quick answer:** A parabola is the set of points equally far from a focus and a directrix. An ellipse has distances to two foci that add to a constant, and a hyperbola has distances to two foci that differ by a constant. Completing the square rewrites any of them in standard form.

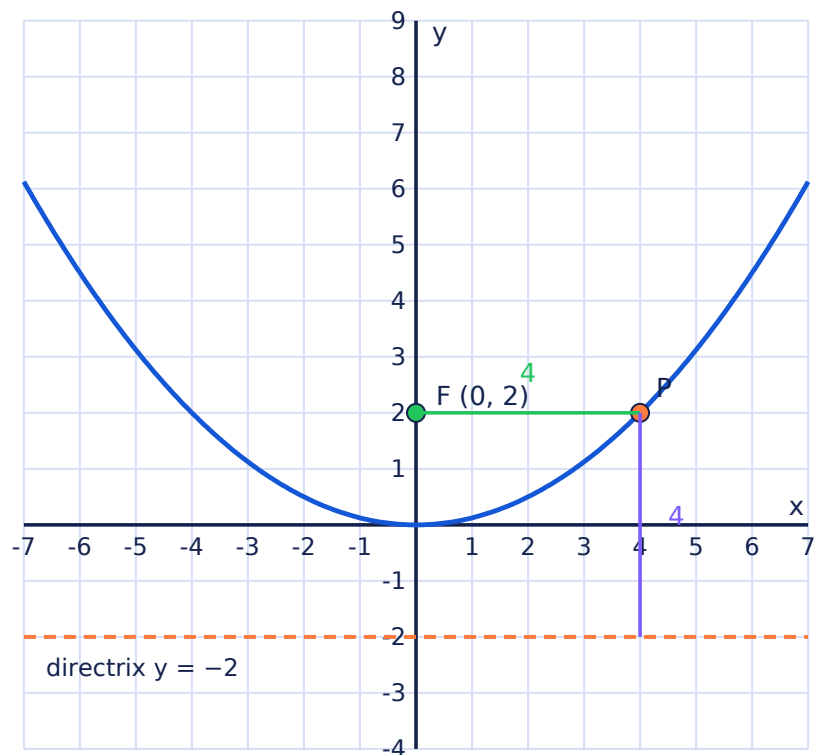
**Standards:** G-GPE.A.1, G-GPE.A.2, G-GPE.A.3 · **Level:** Grade 12 Precalculus

## What is a conic section, and how do the circle and parabola work?

A conic section is a curve made by slicing a double cone with a plane. The four main shapes are the circle, parabola, ellipse and hyperbola. A circle with center  $(h, k)$  and radius  $r$  has equation  $(x - h)^2 + (y - k)^2 = r^2$ .

### Parabola

A parabola is the set of points that are the same distance from a fixed point, the focus, and a fixed line, the directrix. With vertex  $(h, k)$  and focal distance  $p$ , the upward or downward form is  $(x - h)^2 = 4p(y - k)$ , with focus  $(h, k + p)$  and directrix  $y = k - p$ . Replace the roles of  $x$  and  $y$  for a sideways parabola.



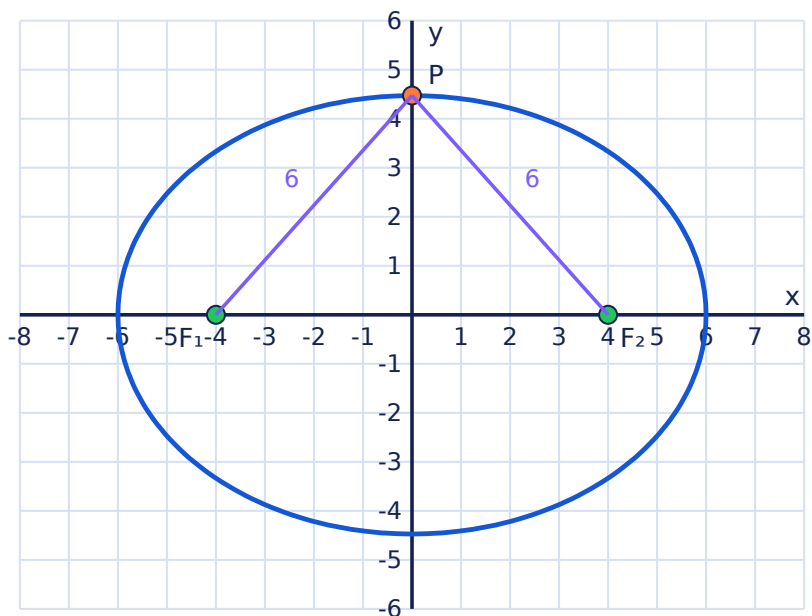
Every point of the parabola is as far from the focus as from the directrix.

## How are ellipses and hyperbolas defined by their foci?

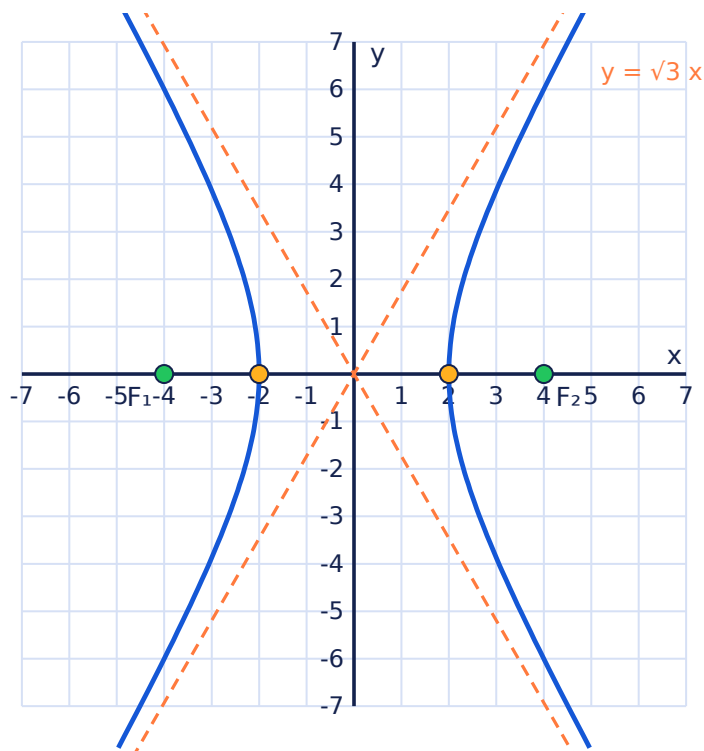
### Ellipse and hyperbola

An ellipse is the set of points whose distances to two foci add to a constant  $2a$ . A hyperbola is the set of points whose distances to two foci differ by a constant  $2a$ . In standard form centered at  $(h, k)$  with a horizontal main axis:

| Conic     | Equation                                        | Foci relation     | Extras                                      |
|-----------|-------------------------------------------------|-------------------|---------------------------------------------|
| Ellipse   | $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$ | $c^2 = a^2 - b^2$ | vertices $(h \pm a, k)$                     |
| Hyperbola | $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$ | $c^2 = a^2 + b^2$ | asymptotes $y - k = \pm \frac{b}{a}(x - h)$ |



The distances from P to the two foci add up to 12, the length of the major axis.



The dashed asymptotes  $y = \pm\sqrt{3}x$  guide the two branches.

### Common mistake

Using  $c^2 = a^2 + b^2$  for an ellipse. In an ellipse,  $a$  is the largest of the three and  $c$  is smaller than  $a$ .

## How do you identify a conic by completing the square?

### Method

1. Group the  $x$  terms and the  $y$  terms, and move the constant to the right.
2. Factor out any leading coefficients, then complete each square (adding the same amount to both sides).
3. Divide to make the right side 1. Two squares with plus signs give an ellipse (or circle); a minus sign gives a hyperbola; a single square gives a parabola.

### Worked example

For  $4x^2 + 9y^2 - 16x + 18y - 11 = 0$ , write  $4(x - 2)^2 + 9(y + 1)^2 = 11 + 16 + 9 = 36$ , so  $\frac{(x - 2)^2}{9} + \frac{(y + 1)^2}{4} = 1$ . It is an ellipse with center  $(2, -1)$ ,  $a = 3$ ,  $b = 2$  and  $c = \sqrt{5}$ .

### Mathbot's tip

Find  $c$  last. First read  $a$  and  $b$  from the denominators, then use the right relationship to locate the foci.

### Key points to remember

- Circle:  $(x - h)^2 + (y - k)^2 = r^2$ ; complete the square to find the center and radius.
- Parabola  $(x - h)^2 = 4p(y - k)$ : focus  $(h, k + p)$ , directrix  $y = k - p$ .
- Ellipse:  $c^2 = a^2 - b^2$ , and the sum of focal distances is  $2a$ .
- Hyperbola:  $c^2 = a^2 + b^2$ , the difference of focal distances is  $2a$ , and the asymptotes have slope  $\pm b/a$ .
- A plus sign between squares means ellipse, a minus sign means hyperbola, one square means parabola.

## Parabolas, Ellipses and Hyperbolas in Standard Form: frequently asked questions

### What is the focus of a parabola?

The focus is the fixed point that defines the parabola together with the directrix. Every point on the curve is exactly as far from the focus as from the directrix. Signals arriving parallel to the axis reflect to the focus, which is why dishes use this shape.

### How do I know whether a conic is an ellipse or a hyperbola?

Compare the signs of the squared terms after completing the square. Same signs give an ellipse (or a circle if the denominators match), and opposite signs give a hyperbola.

### Which denominator is $a^2$ in an ellipse?

The larger denominator is  $a^2$ , and it sits under the variable along the major axis. If the larger denominator is under  $x$ , the major axis is horizontal; if under  $y$ , it is vertical.

### **What do the asymptotes of a hyperbola do?**

They are the lines the branches approach but never touch. For a horizontal hyperbola centered at  $(h, k)$ , they pass through the center with slopes  $\pm b/a$ . Drawing the box with sides  $2a$  and  $2b$  locates them.

### **Why do I need to add to both sides when completing the square?**

Adding a number to only one side changes the equation. Whatever is added to complete a square on the left, multiplied by any factored-out coefficient, must also be added to the right.