

Solving Systems and Transforming the Plane

Grade 12 – Precalculus math · Lesson · Standards: A-REI.C.8, A-REI.C.9, N-VM.C.11, N-VM.C.12

Quick answer: A system of linear equations can be written as $AX = B$, where A holds the coefficients. When $\det A \neq 0$, the unique solution is $X = A^{-1}B$. A 2×2 matrix also transforms the plane, and the absolute value of its determinant is the factor by which areas change.

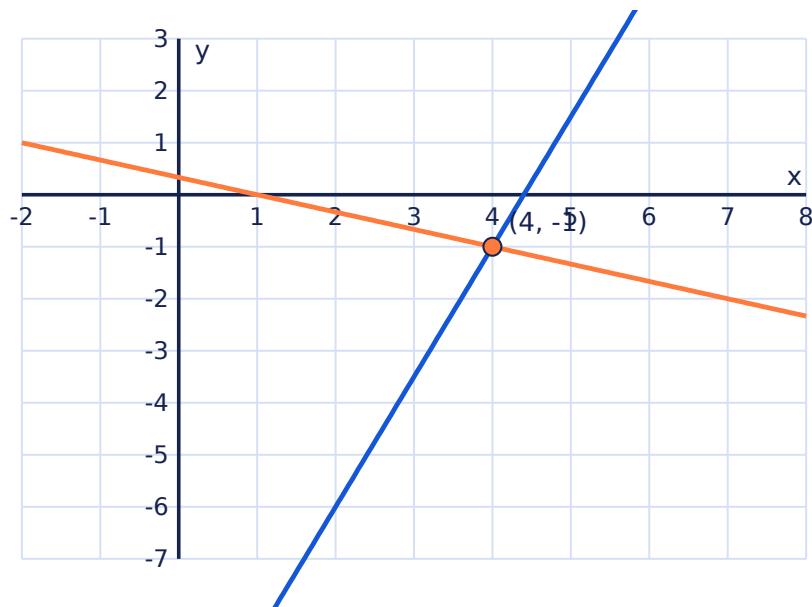
Standards: A-REI.C.8, A-REI.C.9, N-VM.C.11, N-VM.C.12 · **Level:** Grade 12 Precalculus

How do you write a system as a matrix equation?

A pair of linear equations can be stacked into one statement, $AX = B$. The coefficients form the matrix A , the unknowns form a column X , and the constants form a column B . For $5x - 2y = 22$ and $x + 3y = 1$:

Matrix equation

A has rows $(5, -2)$ and $(1, 3)$. X is the column with entries x and y . B is the column with entries 22 and 1. The statement $AX = B$ then says the same thing as both equations together.



The two lines meet at one point, the solution of the system.

How do you solve it with an inverse matrix?

Method: solving $AX = B$

1. Compute $\det A = ad - bc$. If it is 0, there is no unique solution.
2. Build A^{-1} by exchanging the diagonal, flipping the off-diagonal signs and dividing by $\det A$.
3. Multiply on the left: $X = A^{-1}B$.
4. Substitute the answer into both equations.

Worked example

Here $\det A = 5(3) - (-2)(1) = 17$, so $A^{-1} = \frac{1}{17}$ times rows (3, 2) and $(-1, 5)$. Multiplying by the column (22, 1) gives $\frac{1}{17}(3 \cdot 22 + 2 \cdot 1, -22 + 5) = \frac{1}{17}(68, -17) = (4, -1)$. So $x = 4$ and $y = -1$.

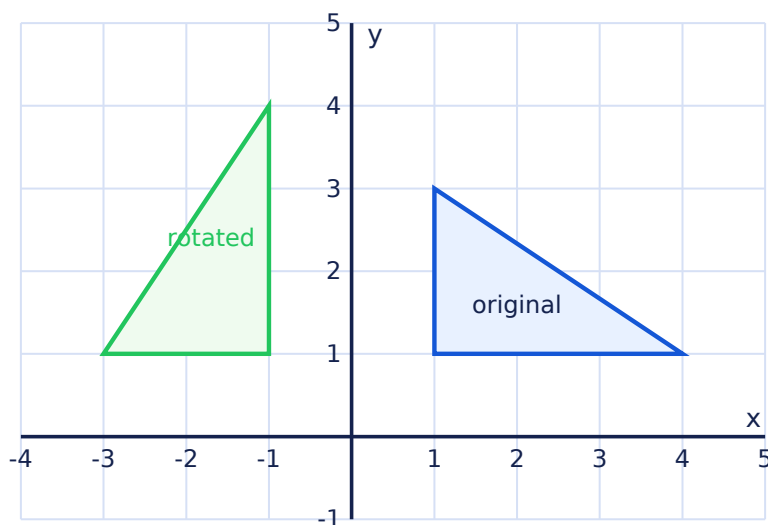
Common mistake

Computing BA^{-1} . Matrix products depend on order, and the inverse must multiply B from the left. For three or more unknowns, use technology to find the inverse.

How do 2×2 matrices transform the plane?

Multiplying a matrix by a point written as a column moves the point. Each column of the matrix tells where the points (1, 0) and (0, 1) land.

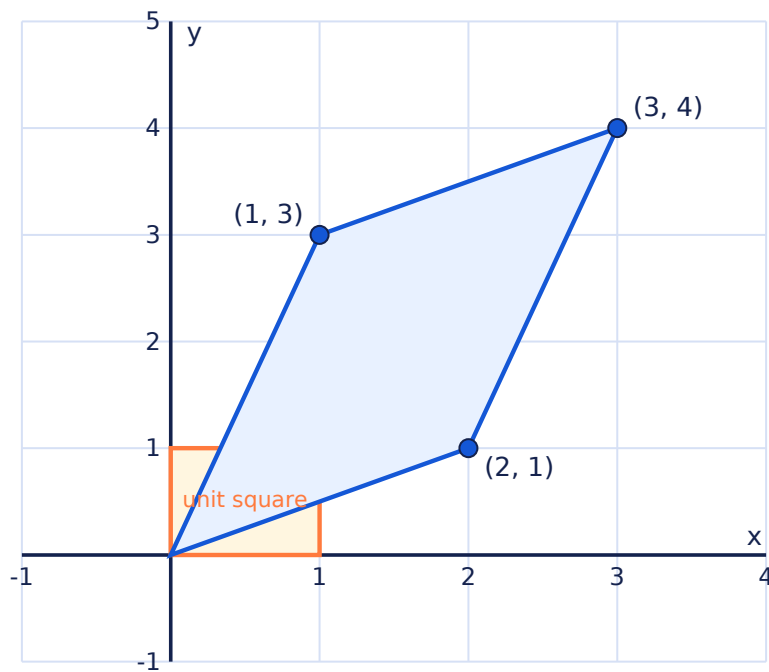
Transformation	Matrix rows	Effect on (x, y)
Rotation 90° counterclockwise	(0, -1) and (1, 0)	$(-y, x)$
Reflection in the x-axis	(1, 0) and (0, -1)	$(x, -y)$
Reflection in the y-axis	$(-1, 0)$ and (0, 1)	$(-x, y)$
Scale by k	$(k, 0)$ and $(0, k)$	(kx, ky)
Horizontal shear	(1, k) and (0, 1)	$(x + ky, y)$



A 90° counterclockwise rotation sends (x, y) to $(-y, x)$; the area stays 3.

What does the determinant tell you about area?

The absolute value of the determinant is the factor by which a transformation scales area. If $\det = 2$, every shape doubles in area. A negative determinant reverses orientation, and $\det = 0$ flattens the plane onto a line. For the matrix with rows $(2, 1)$ and $(1, 3)$, $\det = 5$, so the unit square becomes a parallelogram of area 5.



The matrix with rows $(2, 1)$ and $(1, 3)$ turns the unit square (area 1) into a parallelogram of area 5.

Mathbot's tip

To build a transformation matrix, track where $(1, 0)$ and $(0, 1)$ go and write those images as the two columns.

Key points to remember

- Write a system as $AX = B$: coefficient matrix, unknown column, constant column.
- If $\det A \neq 0$, then $X = A^{-1}B$; multiply on the left. If $\det A = 0$, there is no unique solution.
- A 2×2 matrix times a point written as a column gives the transformed point.
- The columns of a transformation matrix are the images of $(1, 0)$ and $(0, 1)$.
- Areas scale by $|\det|$; a negative determinant also flips orientation.

Solving Systems and Transforming the Plane: frequently asked questions

Why must the inverse multiply from the left?

Because $AX = B$ multiplies X by A on the left. To cancel A , multiply both sides by A^{-1} on the left,

giving $X = A^{-1}B$. Multiplying on the right is not a valid step and usually produces a product that is not even defined.

What if $\det A$ is zero?

Then A has no inverse. The lines are either parallel, so there is no solution, or the same line, so there are infinitely many. Row-reducing or substituting tells which case you have.

How do I solve a 3×3 system with matrices?

Write $AX = B$ with a 3×3 matrix A , then use a calculator or software to compute A^{-1} and multiply by B . The idea is identical to the 2×2 case; only the arithmetic grows.

How do I find the matrix of a transformation?

Apply the transformation to $(1, 0)$ and $(0, 1)$. Write the image of $(1, 0)$ as the first column and the image of $(0, 1)$ as the second column. Every other point follows by linearity.

What does a negative determinant mean for a transformation?

The area still scales by the absolute value of the determinant, but the shape is flipped like a mirror image, as with a reflection. A reflection has determinant -1 .