

# Solving Systems and Transforming the Plane

Grade 12 – Precalculus math · Worksheet Answer Key · Standards: A-REI.C.8, A-REI.C.9, N-VM.C.11, N-VM.C.12

Each answer shows the determinant, the inverse, the multiplication, and a check by substitution. When a transformation is involved, the answer also compares the determinant with the area directly.

**Standards:** A-REI.C.8, A-REI.C.9, N-VM.C.11, N-VM.C.12 · **Level:** Grade 12 Precalculus

## Problem 1 Writing a Matrix Equation

The coefficients form  $A$ , the unknowns form  $X$  and the constants form  $B$ :

|   |    |
|---|----|
| 4 | -1 |
| 2 | 5  |

$X$  is the column with entries  $x$  and  $y$ , and  $B$  is the column with entries 9 and  $-3$ .

Answer:  **$A$  has rows  $(4, -1)$  and  $(2, 5)$ ,  $X = (x, y)$  as a column,  $B = (9, -3)$  as a column**

## Problem 2 Reading a Matrix Equation

The first row gives  $2x - 3y = 8$  and the second row gives  $5x + y = 3$ .

Test  $(1, -2)$ :  $2(1) - 3(-2) = 2 + 6 = 8$  and  $5(1) + (-2) = 3$ . Both hold.

Answer:  **$2x - 3y = 8$  and  $5x + y = 3$ ; yes,  $(1, -2)$  is a solution**

## Problem 3 Is There One Solution?

$A$  has rows  $(6, 9)$  and  $(2, 3)$ .  $\det A = 6(3) - 9(2) = 18 - 18 = 0$ .

A zero determinant means no unique solution. The left sides are proportional (the first is 3 times the second) but  $5 \neq 3 \cdot 7$ , so the lines are parallel and there is no solution.

Answer:  **$\det = 0$ , so there is no unique solution (the lines are parallel)**

## Problem 4 Matrix Times a Point

First entry:  $3(5) + (-1)(-2) = 15 + 2 = 17$ . Second entry:  $2(5) + 4(-2) = 10 - 8 = 2$ .

Answer: **the column (17, 2)**

### Problem 5 Reflecting a Point

First entry:  $-1(6) + 0(-4) = -6$ . Second entry:  $0(6) + 1(-4) = -4$ .

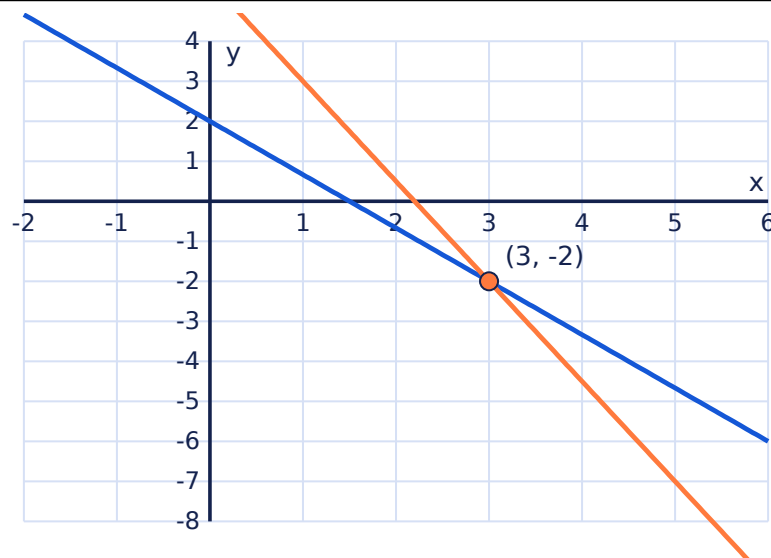
Answer: **(-6, -4)**

### Problem 6 Stretching the Unit Square

$\det = 3(2) - 0(0) = 6$ . The image is a 3 by 2 rectangle with area 6 square units, which equals  $|\det|$ .

Answer:  **$\det = 6$ ; area 6 square units**

### Problem 7 Solving with an Inverse



*The solution is the crossing point.*

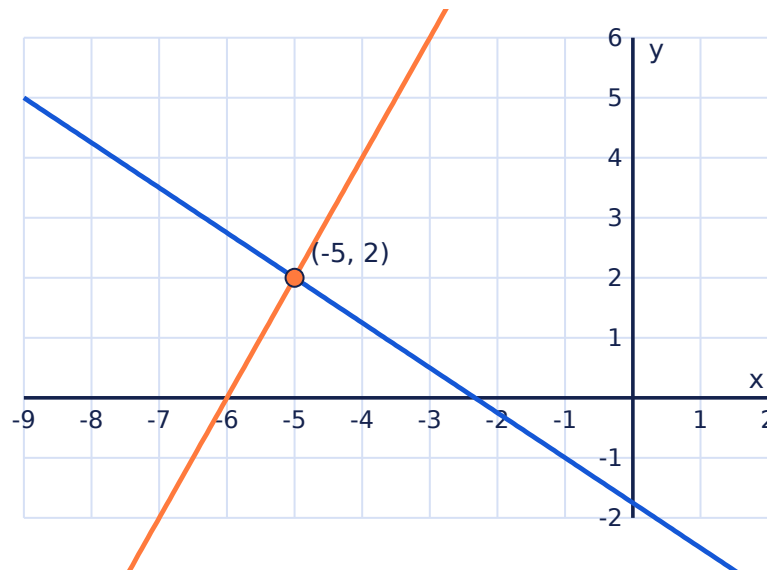
A has rows (4, 3) and (5, 2);  $\det A = 4(2) - 3(5) = -7$ . So  $A^{-1} = -\frac{1}{7}$  times rows (2, -3) and (-5, 4).

$$X = A^{-1}B = -\frac{1}{7}(2 \cdot 6 - 3 \cdot 11, -5 \cdot 6 + 4 \cdot 11) = -\frac{1}{7}(-21, 14) = (3, -2).$$

Check:  $4(3) + 3(-2) = 6$  and  $5(3) + 2(-2) = 11$ .

Answer:  **$x = 3$  and  $y = -2$**

### Problem 8 A Negative Solution



The solution is the crossing point.

$\det A = 3(-1) - 4(2) = -11$ .  $A^{-1} = -\frac{1}{11}$  times rows  $(-1, -4)$  and  $(-2, 3)$ .

$X = -\frac{1}{11}((-1)(-7) + (-4)(-12), (-2)(-7) + 3(-12)) = -\frac{1}{11}(55, -22) = (-5, 2)$ .

Check:  $3(-5) + 4(2) = -7$  and  $2(-5) - 2 = -12$ .

Answer:  **$x = -5$  and  $y = 2$**

### Problem 9 A Fractional Solution

$\det A = 2(-1) - 3(4) = -14$ .  $A^{-1} = -\frac{1}{14}$  times rows  $(-1, -3)$  and  $(-4, 2)$ .

$X = -\frac{1}{14}(-5 - 18, -20 + 12) = -\frac{1}{14}(-23, -8) = (\frac{23}{14}, \frac{4}{7})$ .

Check:  $2(\frac{23}{14}) + 3(\frac{4}{7}) = \frac{23}{7} + \frac{12}{7} = 5$  and  $4(\frac{23}{14}) - \frac{4}{7} = \frac{46}{7} - \frac{4}{7} = 6$ .

Answer:  **$x = \frac{23}{14}$  and  $y = \frac{4}{7}$**

### Problem 10 Two Moves in a Row

Rotation:  $(0)(-3) + (-1)(5) = -5$  and  $(1)(-3) + 0(5) = -3$ , giving  $(-5, -3)$ . Reflection in the  $x$ -axis changes the sign of  $y$ :  $(-5, 3)$ .

The combined matrix is the reflection matrix times the rotation matrix, in that order:

|    |    |
|----|----|
| 0  | -1 |
| -1 | 0  |

Applying it to  $(-3, 5)$ :  $0(-3) - 1(5) = -5$  and  $-1(-3) + 0(5) = 3$ , which matches.

Answer:  **$(-5, 3)$ ; combined matrix with rows  $(0, -1)$  and  $(-1, 0)$**

### Problem 11 Area After a Transformation

---

$\det = 4(3) - (-2)(1) = 12 + 2 = 14$ . Areas are multiplied by  $|\det| = 14$ .

New area  $= 6 \cdot 14 = 84$  square units.

Answer: **84 square units**

### Problem 12 Building a Scaling Matrix

---

The images of  $(1, 0)$  and  $(0, 1)$  are  $(2, 0)$  and  $(0, 3)$ , so the matrix has rows  $(2, 0)$  and  $(0, 3)$ .

Image of  $(2, -1)$ :  $(2 \cdot 2 + 0, 0 + 3(-1)) = (4, -3)$ . Determinant  $= 6$ , so the unit square becomes a rectangle of area 6.

Answer: **matrix with rows  $(2, 0)$ ,  $(0, 3)$ ; image  $(4, -3)$ ; area 6**

### Problem 13 A Matrix That Flattens

---

$\det = 2(6) - 4(3) = 0$ . The image of  $(1, 0)$  is  $(2, 3)$  and the image of  $(0, 1)$  is  $(4, 6)$ .

Both images lie on the line  $y = 1.5x$ . Every point is sent to that line, so the plane collapses onto a line and areas become 0. The matrix has no inverse.

Answer:  **$\det = 0$ ; the plane collapses onto the line  $y = 1.5x$**

### Problem 14 A Shear

---

$(0, 0)$  stays  $(0, 0)$ .  $(3, 0)$  goes to  $(3 + 0, 0) = (3, 0)$ .  $(0, 2)$  goes to  $(0 + 2 \cdot 2, 2) = (4, 2)$ .

The original area is  $\frac{1}{2}(3)(2) = 3$ . The image has base 3 and height 2, so its area is also 3. This agrees with  $\det = 1(1) - 2(0) = 1$ .

Answer: **images  $(0, 0)$ ,  $(3, 0)$ ,  $(4, 2)$ ; both areas equal 3**

### Problem 15 Food Truck Prices

---

Let  $w$  and  $b$  be the prices.  $A$  has rows  $(18, 12)$  and  $(25, 20)$ , and  $B$  is the column  $(246, 365)$ .

$\det A = 18(20) - 12(25) = 360 - 300 = 60$ , so  $A^{-1} = \frac{1}{60}$  times rows  $(20, -12)$  and  $(-25, 18)$ .

$$w = \frac{(20 \cdot 246 - 12 \cdot 365)}{60} = \frac{540}{60} = 9 \text{ and } b = \frac{(-25 \cdot 246 + 18 \cdot 365)}{60} = \frac{420}{60} = 7.$$

Check:  $18(9) + 12(7) = 246$  and  $25(9) + 20(7) = 365$ .

Answer: **a wrap costs \$9 and a bowl costs \$7**

### Problem 16 Mixing Solutions

Total volume:  $x + y = 12$ . Acid:  $0.2x + 0.5y = 0.38(12) = 4.56$ .  $A$  has rows  $(1, 1)$  and  $(0.2, 0.5)$ ,  $B = (12, 4.56)$ .

$\det A = 1(0.5) - 1(0.2) = 0.3$ .  $A^{-1} = \frac{1}{0.3}$  times rows  $(0.5, -1)$  and  $(-0.2, 1)$ .

$x = \frac{(0.5(12) - 4.56)}{0.3} = \frac{1.44}{0.3} = 4.8$  and  $y = \frac{(-0.2(12) + 4.56)}{0.3} = \frac{2.16}{0.3} = 7.2$ .

Check:  $4.8 + 7.2 = 12$  and  $0.2(4.8) + 0.5(7.2) = 0.96 + 3.6 = 4.56$ .

Answer: **4.8 liters of the 20% solution and 7.2 liters of the 50% solution**

### Problem 17 Theater Ticket Prices

Dividing each equation by 5 gives  $8a + 6c + 2s = 162$ ,  $5a + 9c + 4s = 168$  and  $6a + 4c + 6s = 158$ .  $A$  has rows  $(8, 6, 2)$ ,  $(5, 9, 4)$ ,  $(6, 4, 6)$  and  $B$  is the column  $(162, 168, 158)$ .

Technology gives  $\det A = 200$  and  $A^{-1} = \frac{1}{200}$  times the matrix:

|     |     |     |
|-----|-----|-----|
| 38  | -28 | 6   |
| -6  | 36  | -22 |
| -34 | 4   | 42  |

Multiplying by  $B$  gives the column  $(12, 8, 9)$ .

Check:  $40(12) + 30(8) + 10(9) = 810$ ;  $25(12) + 45(8) + 20(9) = 840$ ;  $30(12) + 20(8) + 30(9) = 790$ .

Answer: **adult \$12, child \$8, senior \$9**

### Problem 18 Robot Arm Zone

$(2, 1)$  goes to  $(0(2) - 2.5(1), 2.5(2) + 0) = (-2.5, 5)$ .  $(6, 1)$  goes to  $(-2.5, 15)$ .  $(2, 4)$  goes to  $(-10, 5)$ .

Original area:  $\frac{1}{2}(4)(3) = 6$ .  $\det = 0(0) - (-2.5)(2.5) = 6.25$ , so the new area is  $6(6.25) = 37.5$ .

Direct check: the new triangle has a vertical side from  $(-2.5, 5)$  to  $(-2.5, 15)$  of length 10 and a horizontal distance of 7.5 to the vertex  $(-10, 5)$ , so its area is  $\frac{1}{2}(10)(7.5) = 37.5$ .

Answer: **vertices  $(-2.5, 5)$ ,  $(-2.5, 15)$ ,  $(-10, 5)$ ; area 37.5 square units**

### Problem 19 Transformation from Images

(a) The images become the columns:  $M$  has rows  $(3, -1)$  and  $(1, 2)$ .

(b)  $(3(2) - 1(5), 1(2) + 2(5)) = (1, 12)$ .

(c)  $\det = 3(2) - (-1)(1) = 7$ . Areas are multiplied by 7, and orientation is preserved because  $\det > 0$ .

(d)  $M^{-1} = \frac{1}{7}$  times rows  $(2, 1)$  and  $(-1, 3)$ . Applying it to  $(8, 5)$ :  $\frac{1}{7}(16 + 5, -8 + 15) = \frac{1}{7}(21, 7) = (3, 1)$ . Check:  $M(3, 1) = (9 - 1, 3 + 2) = (8, 5)$ .

Answer: **(a) rows  $(3, -1)$ ,  $(1, 2)$ ; (b)  $(1, 12)$ ; (c)  $\det = 7$ ; (d)  $(3, 1)$**

### Problem 20 When Is There No Unique Solution?

(a)  $\det = 2(6) - k(3) = 12 - 3k = 0$  gives  $k = 4$ .

(b) With  $k = 4$ :  $2x + 4y = 8$  simplifies to  $x + 2y = 4$ , and  $3x + 6y = 12$  simplifies to the same equation. The lines coincide, so there are infinitely many solutions.

(c) For  $k = 5$ ,  $\det = 12 - 15 = -3$  and  $A^{-1} = -\frac{1}{3}$  times rows  $(6, -5)$  and  $(-3, 2)$ .  $X = -\frac{1}{3}(6 \cdot 8 - 5 \cdot 12, -3 \cdot 8 + 2 \cdot 12) = -\frac{1}{3}(-12, 0) = (4, 0)$ .

Check:  $2(4) + 5(0) = 8$  and  $3(4) + 6(0) = 12$ .

Answer: **(a)  $k = 4$ ; (b) infinitely many solutions; (c)  $(4, 0)$**