

# Polar Coordinates and Parametric Curves

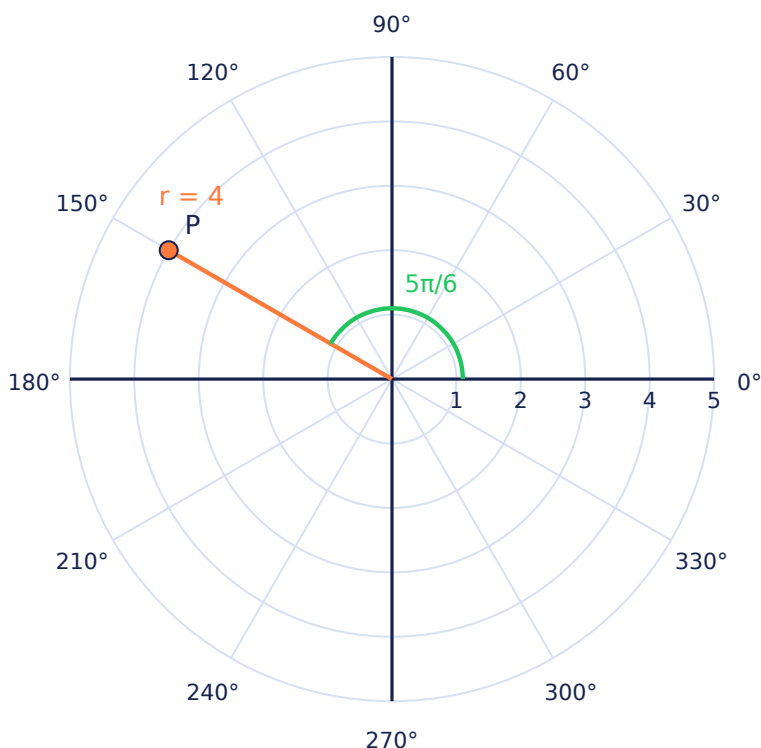
Grade 12 – Precalculus math · Lesson · Standards: N-CN.B.4, F-IF.C.7

**Quick answer:** Polar coordinates  $(r, \theta)$  name a point by its distance from the origin and its angle from the positive x-axis, with  $x = r \cos \theta$  and  $y = r \sin \theta$ . Parametric equations give  $x$  and  $y$  as functions of a parameter such as time, and eliminating the parameter reveals the path's rectangular equation.

**Standards:** N-CN.B.4, F-IF.C.7 · **Level:** Grade 12 Precalculus

## How do polar coordinates locate a point?

A point is named by a distance and a direction. In polar coordinates  $(r, \theta)$ ,  $r$  is the signed distance from the pole (the origin) and  $\theta$  is the angle from the polar axis (the positive x-axis). A negative  $r$  means "go the opposite way", so  $(-3, 40^\circ)$  and  $(3, 220^\circ)$  are the same point, and adding  $360^\circ$  or  $2\pi$  to  $\theta$  changes nothing.



*P has polar coordinates  $(4, 5\pi/6)$ , which is  $(-2\sqrt{3}, 2)$  in rectangular form.*

## How do you convert between polar and rectangular form?

### Conversion formulas

$$x = r \cos \theta, y = r \sin \theta, r^2 = x^2 + y^2, \text{ and } \tan \theta = \frac{y}{x} \text{ for } x \neq 0.$$

### Method: converting an equation

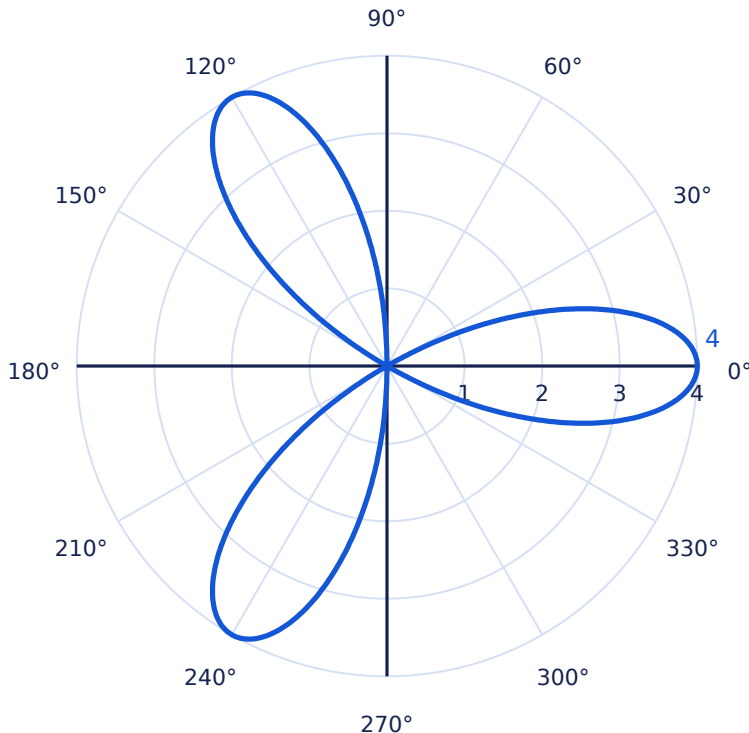
1. Multiply both sides by  $r$  if that creates an  $r \cos \theta$  or  $r \sin \theta$  term.
2. Replace  $r \cos \theta$  with  $x$ ,  $r \sin \theta$  with  $y$ , and  $r^2$  with  $x^2 + y^2$ .
3. Complete the square to name the shape.

### Worked example

Convert  $r = 6 \cos \theta$ . Multiply by  $r$ :  $r^2 = 6r \cos \theta$ , so  $x^2 + y^2 = 6x$ . Completing the square gives  $(x - 3)^2 + y^2 = 9$ , a circle with center  $(3, 0)$  and radius 3.

## Which shapes do the common polar equations make?

| Equation                                 | Graph   | Key feature                                 |
|------------------------------------------|---------|---------------------------------------------|
| $r = a$                                  | circle  | center at the pole, radius $ a $            |
| $r = a \cos \theta$ or $a \sin \theta$   | circle  | passes through the pole, diameter $ a $     |
| $r = a \cos n\theta$ or $a \sin n\theta$ | rose    | $n$ petals if $n$ is odd, $2n$ if even      |
| $r = a + b \cos \theta$                  | limaçon | inner loop if $a < b$ ; cardioid if $a = b$ |



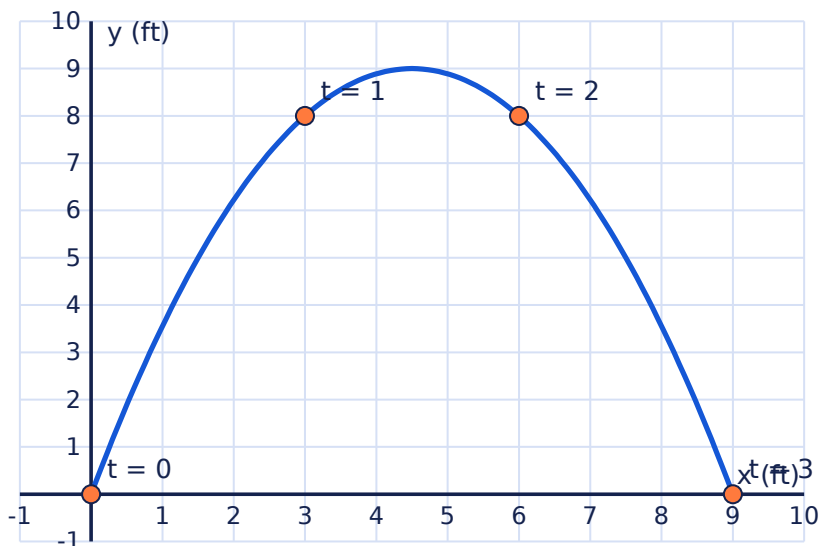
The rose  $r = 4 \cos 3\theta$  has 3 petals, each 4 units long.

### Common mistake

Using  $\tan^{-1}(y/x)$  without checking the quadrant. For  $(-2, 2)$  the calculator gives  $-45^\circ$ , but the point is at  $135^\circ$ .

## What are parametric equations and how do you eliminate the parameter?

Parametric equations write both coordinates as functions of a third variable, usually time  $t$ :  $x = f(t)$  and  $y = g(t)$ . To eliminate the parameter, solve one equation for  $t$  and substitute it into the other. For  $x = 3t$  and  $y = 12t - 4t^2$ ,  $t = x/3$ , so  $y = 4x - \frac{4x^2}{9}$ , a parabola.



$x = 3t$  and  $y = 12t - 4t^2$ : the parameter  $t$  tells where the drone is at each second.

### Mathbot's tip

For a trigonometric pair such as  $x = 4 \cos t$  and  $y = 2 \sin t$ , do not solve for  $t$ . Use  $\cos^2 t + \sin^2 t = 1$  instead.

### Key points to remember

- Conversion:  $x = r \cos \theta$ ,  $y = r \sin \theta$ ,  $r^2 = x^2 + y^2$ .
- A point has many polar names:  $(-r, \theta + \pi)$  and  $(r, \theta + 2\pi)$  name the same point as  $(r, \theta)$ .
- Roses  $r = a \cos n\theta$  have  $n$  petals when  $n$  is odd and  $2n$  when  $n$  is even.
- Limaçons  $r = a + b \cos \theta$  have an inner loop when  $a < b$  and a cusp when  $a = b$ .
- To eliminate a parameter, solve for  $t$  or use an identity such as  $\cos^2 t + \sin^2 t = 1$ .

## Polar Coordinates and Parametric Curves: frequently asked questions

### Can $r$ be negative in polar coordinates?

Yes. A negative  $r$  means moving that distance in the direction opposite to  $\theta$ . So  $(-5, 30^\circ)$  is the same point as  $(5, 210^\circ)$ . Always check the sign when reading a point off a graph.

### **Why does $r = 6 \cos \theta$ make a circle?**

Multiplying by  $r$  gives  $x^2 + y^2 = 6x$ , which becomes  $(x - 3)^2 + y^2 = 9$  after completing the square. The curve passes through the pole and has diameter 6.

### **How many petals does a rose with an even $n$ have?**

A rose  $r = a \cos n\theta$  or  $a \sin n\theta$  has  $2n$  petals when  $n$  is even. So  $r = 5 \sin 4\theta$  has 8 petals, each of length 5.

### **How do I eliminate the parameter from $x = \cos t$ and $y = \sin t$ ?**

Square both equations and add:  $x^2 + y^2 = \cos^2 t + \sin^2 t = 1$ . The path is the unit circle, which a single equation in  $x$  and  $y$  shows.

### **Does the rectangular equation show the direction of motion?**

No. Eliminating the parameter loses the timing and direction. Keep a table of  $t$  values or arrows on the graph when the order of travel matters, such as when a path is traced only once.