

Polynomial and Rational Functions: Zeros and Asymptotes

Grade 12 – Precalculus math · Lesson · Standards: A-APR.B.2, A-APR.B.3, F-IF.C.7c, F-IF.C.7d, N-CN.C.9

Quick answer: Every polynomial of degree n has exactly n complex zeros, found by listing candidates p/q with the Rational Root Theorem and testing them. Rational functions can have vertical, horizontal or slant asymptotes, and sign charts built from zeros and undefined points solve inequalities.

Standards: A-APR.B.2, A-APR.B.3, F-IF.C.7c, F-IF.C.7d, N-CN.C.9 · **Level:** Grade 12 Precalculus

How do you find all the zeros of a polynomial?

The Fundamental Theorem of Algebra says a polynomial of degree n has exactly n complex zeros, counting multiplicity. When all coefficients are real, non-real zeros arrive in conjugate pairs: if $a + bi$ is a zero, so is $a - bi$.

Rational Root Theorem

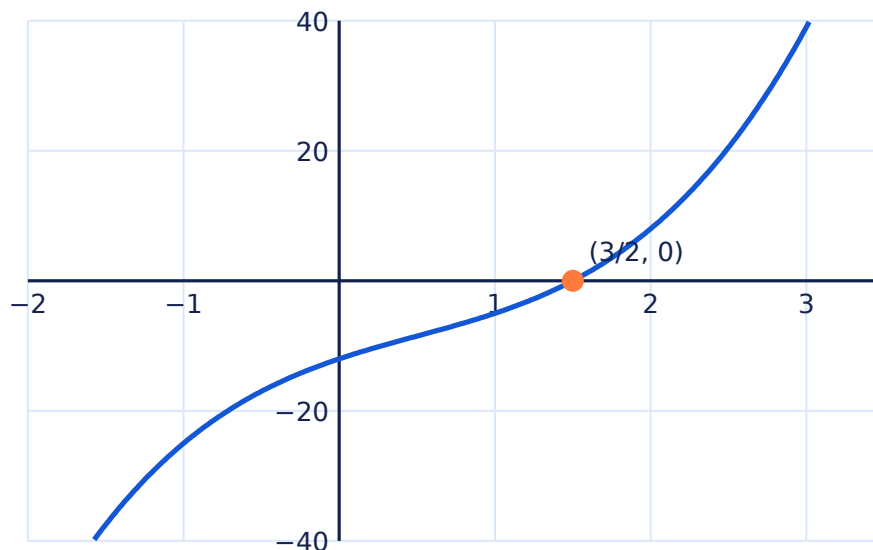
If a polynomial with integer coefficients has a rational zero p/q in lowest terms, then p divides the constant term and q divides the leading coefficient.

Method: find every zero

1. List the candidates p/q , both signs.
2. Test candidates with synthetic division until the remainder is 0.
3. Factor the quotient (quadratic formula if needed) to get the remaining zeros, real or complex.

Worked example

$P(x) = 2x^3 - 3x^2 + 8x - 12$. Candidates have p from $\pm 1, 2, 3, 4, 6, 12$ and q from $1, 2$. Testing $3/2$ gives remainder 0, and the quotient is $2x^2 + 8$. Then $P(x) = (2x - 3)(x^2 + 4)$, so the zeros are **$3/2$, $2i$ and $-2i$** .



The graph of P crosses the x -axis only once, at $x = 3/2$. The other two zeros, $\pm 2i$, are not real.

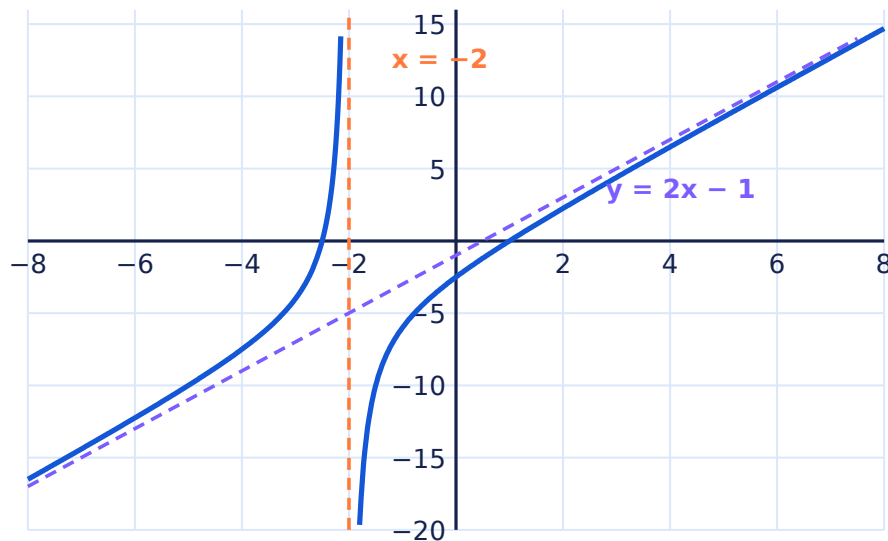
How does the leading term control end behavior?

For large $|x|$, only the leading term $a_n x^n$ matters.

Degree	Leading coefficient	Left end	Right end
even	positive	up	up
even	negative	down	down
odd	positive	down	up
odd	negative	up	down

When does a rational function have a slant asymptote?

When the top degree exceeds the bottom degree by exactly 1, divide. The quotient, with the remainder dropped, gives the line. For $f(x) = \frac{(2x^2 + 3x - 5)}{(x + 2)}$, long division gives $2x - 1$ with remainder -3 , the line is **$y = 2x - 1$** .



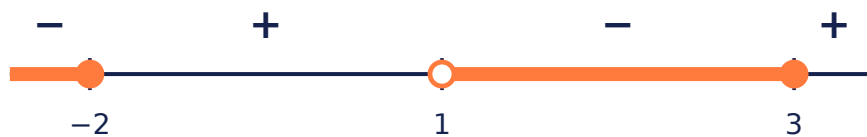
Slant asymptote $y = 2x - 1$ and vertical asymptote $x = -2$.

How do you solve polynomial and rational inequalities?

Move everything to one side, factor, and label every critical number, meaning each zero of the top and each zero of the bottom. Test one value in each interval and record its sign. Zeros of the denominator are never included.

Worked example

Solve $\frac{(x+2)(x-3)}{(x-1)} \leq 0$. The critical numbers are -2 , 1 and 3 , and the signs of the expression are $-$, $+$, $-$, $+$ from left to right. Include -2 and 3 , exclude 1 : $(-\infty, -2] \cup (1, 3]$.



Shaded parts: $(-\infty, -2]$ and $(1, 3]$.

Common mistake

Multiplying both sides of a rational inequality by the denominator. Its sign is unknown, so the inequality might flip. Use a sign chart instead.

Mathbot's tip

Write each critical number as a closed or open dot before shading. Closed dots come from numerators, open dots from denominators.

Key points to remember

- Rational candidates are p/q : p divides the constant term, q divides the leading coefficient.
- A real-coefficient polynomial has non-real zeros in conjugate pairs.
- The leading term decides end behavior: even or odd degree, positive or negative coefficient.
- A slant asymptote appears when the numerator's degree is one more than the denominator's.
- In a sign chart, zeros of the denominator are never included in the solution.

Polynomial and Rational Functions: Zeros and Asymptotes: frequently asked questions

What does the Rational Root Theorem tell you?

It narrows the possible rational zeros to a finite list of fractions p/q . It does not promise that any candidate works, and it says nothing about irrational or complex zeros.

How many zeros does a polynomial have?

A polynomial of degree n has exactly n complex zeros when each zero is counted as many times as its multiplicity. Its graph may show fewer x -intercepts, because some zeros can be non-real.

Why do complex zeros come in conjugate pairs?

If the coefficients are real, substituting $a + bi$ and $a - bi$ into the polynomial gives conjugate results. So when one is a zero, the other must also make the polynomial equal 0.

When does a rational function have a slant asymptote?

It has one when the degree of the numerator is exactly one more than the degree of the denominator. The quotient from polynomial long division, ignoring the remainder, is the equation of the line.

Why use a sign chart instead of multiplying by the denominator?

The denominator can be positive or negative, so multiplying might require flipping the inequality in some intervals and not others. A sign chart tracks each interval separately and avoids that error.