

Series, Binomial Expansions and Limits

Grade 12 – Precalculus math · Lesson · Standards: A-APR.C.5, A-SSE.B.4, F-BF.A.2

Quick answer: A series adds the terms of a sequence, written with sigma notation. Arithmetic series use $S = n(a_1 + a_n)/2$, geometric series use $a_1(1 - r^n)/(1 - r)$, and an infinite geometric series sums to $a_1/(1 - r)$ when r is less than 1 in absolute value. The binomial theorem expands powers, and limits describe values approached.

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What is sigma notation, and how do arithmetic and geometric series work?

Sigma notation adds terms by a rule. $\sum_{k=1}^n a_k$ means $a_1 + a_2 + \dots + a_n$. An arithmetic sequence adds a constant difference d , and a geometric sequence multiplies by a constant ratio r .

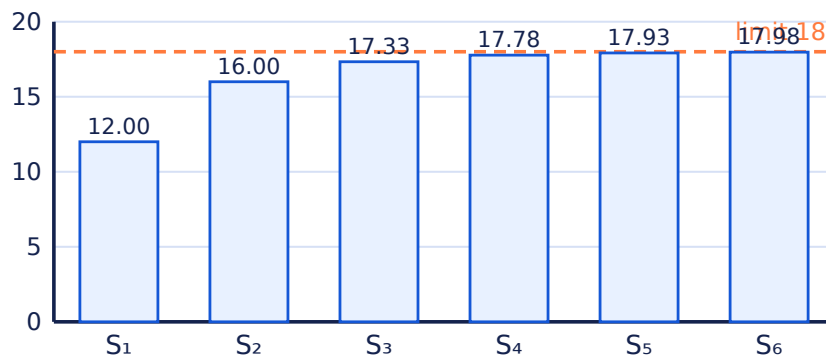
Series	Sum of the first n terms	Condition
Arithmetic	$S_n = \frac{n}{2}(a_1 + a_n)$	none
Geometric	$S_n = a_1 \cdot \frac{(1 - r^n)}{(1 - r)}$	$r \neq 1$
Infinite geometric	$S = \frac{a_1}{(1 - r)}$	$ r < 1$

Worked example

For $\sum_{k=1}^{20} (3k + 2)$, the first term is 5 and the twentieth is 62. The sum is $\frac{20}{2}(5 + 62) = 670$.

When does an infinite geometric series have a sum?

If $|r| < 1$, the partial sums settle toward a single number. For $12 + 4 + \frac{4}{3} + \dots$, $r = \frac{1}{3}$ and the sum is $\frac{12}{1 - \frac{1}{3}} = 18$. If $|r| \geq 1$, the series diverges and has no sum.



Partial sums of the series.

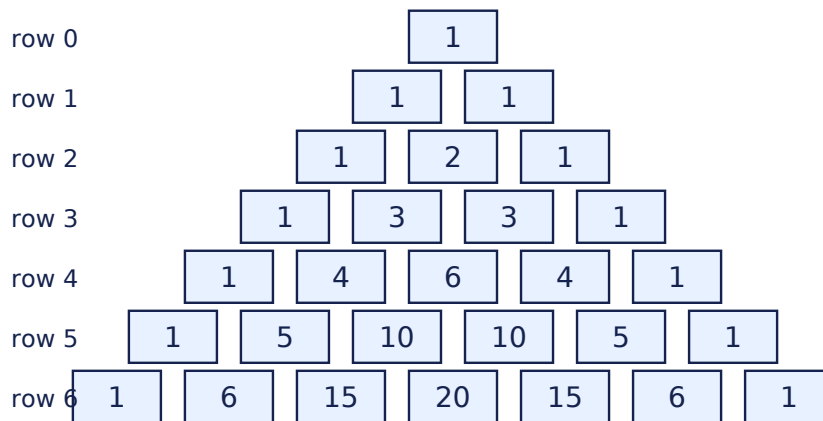
Common mistake

Applying $\frac{a}{1-r}$ when $|r| \geq 1$. Always check the ratio first.

How does the binomial theorem expand a power?

Binomial Theorem

$(a + b)^n = \sum_{k=0}^n C(n, k) a^{n-k} b^k$, where $C(n, k)$ is the entry in row n , position k of Pascal's triangle.

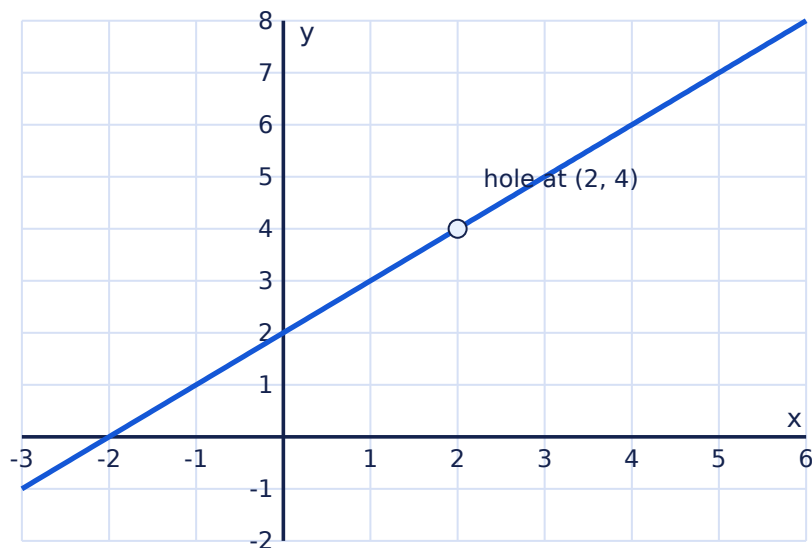


Pascal triangle, rows 0 to 6.

For $(x + 3)^4$ use row 4, which is 1, 4, 6, 4, 1: $x^4 + 4(3)x^3 + 6(9)x^2 + 4(27)x + 81 = x^4 + 12x^3 + 54x^2 + 108x + 81$.

What does a limit describe?

A limit asks what value a function or sequence approaches, not what it equals. The function $f(x) = \frac{x^2 - 4}{x - 2}$ is undefined at $x = 2$, but for $x \neq 2$ it equals $x + 2$. So $\lim_{x \rightarrow 2} f(x) = 4$.



The graph has a hole at (2, 4), yet the outputs approach 4.

Mathbot's tip

For a limit that gives $0/0$, factor and cancel first. The hole in the graph is where the cancelled factor was.

Key points to remember

- Arithmetic: $S_n = n(a_1 + a_n)/2$. Geometric: $S_n = a_1(1 - r^n)/(1 - r)$.
- An infinite geometric series converges to $a_1/(1 - r)$ only if $|r| < 1$.
- Binomial coefficients $C(n, k)$ come from row n of Pascal's triangle.
- A limit describes the value approached; the function need not be defined there.
- For a $0/0$ limit, factor and cancel, then substitute.

Series, Binomial Expansions and Limits: frequently asked questions

What does the index k do in sigma notation?

It counts through the terms. The numbers below and above the sigma give the first and last values of k , and the expression after the sigma is evaluated for each value and added.

How do I know whether an infinite series has a sum?

Find the common ratio. If its absolute value is less than 1, the partial sums approach $a_1/(1 - r)$. Otherwise the sums grow without settling and the series diverges.

How can a limit exist when the function is undefined?

A limit looks at nearby inputs, not the input itself. After canceling a common factor, the function can approach a clear value even though the original expression has a hole at that point.

Why do binomial coefficients match Pascal's triangle?

Each entry is the sum of the two entries above it, which mirrors how terms combine when one more factor of $(a + b)$ is multiplied in. The entry in row n and position k equals $C(n, k)$.

What is the limit of a sequence?

It is the single value the terms get and stay close to as n grows. For example, the terms of $(3n + 1)/(n + 2)$ approach 3 as n becomes very large.