

Proving and Using Trigonometric Identities

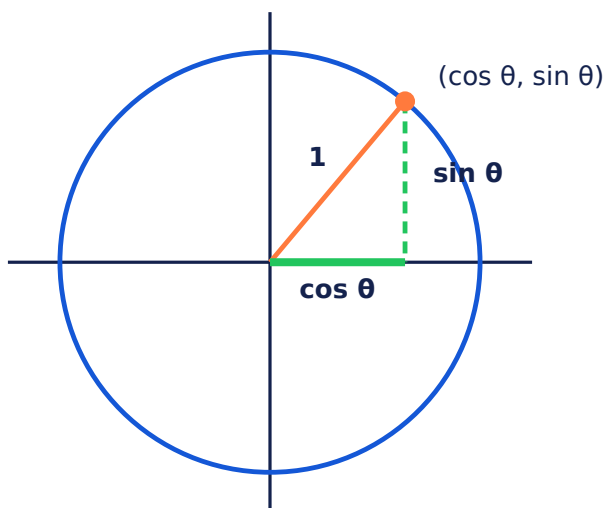
Grade 12 – Precalculus math · Lesson · Standards: F-TF.C.8, F-TF.C.9

Quick answer: A trigonometric identity is an equation true for every allowed angle. The reciprocal, quotient and Pythagorean identities come from the unit circle, and the sum, difference and double-angle formulas let you find exact values such as $\sin 75^\circ$ and rewrite expressions to prove other identities.

Standards: F-TF.C.8, F-TF.C.9 · **Level:** Grade 12 Precalculus

What are the basic trigonometric identities?

An identity is an equation that is true for every input where both sides are defined. The point $(\cos \theta, \sin \theta)$ lies on the unit circle, so its coordinates satisfy $x^2 + y^2 = 1$.



The right triangle in the unit circle gives $\sin^2 \theta + \cos^2 \theta = 1$.

Type	Identities
Reciprocal	$\csc \theta = 1/\sin \theta$, $\sec \theta = 1/\cos \theta$, $\cot \theta = 1/\tan \theta$
Quotient	$\tan \theta = \sin \theta / \cos \theta$, $\cot \theta = \cos \theta / \sin \theta$
Pythagorean	$\sin^2 \theta + \cos^2 \theta = 1$; $1 + \tan^2 \theta = \sec^2 \theta$; $1 + \cot^2 \theta = \csc^2 \theta$

How do you verify an identity?

Method: prove one side equals the other

1. Pick the more complicated side and leave the other side alone.

2. Rewrite everything in terms of sine and cosine.
3. Combine fractions over a common denominator, expand or factor, and use a Pythagorean identity.
4. Stop when the expression matches the other side.

Worked example

Verify $\tan x + \cot x = \sec x \csc x$. Left side: $\sin x / \cos x + \cos x / \sin x = (\sin^2 x + \cos^2 x) / (\sin x \cos x) = 1 / (\sin x \cos x) = \sec x \csc x$.

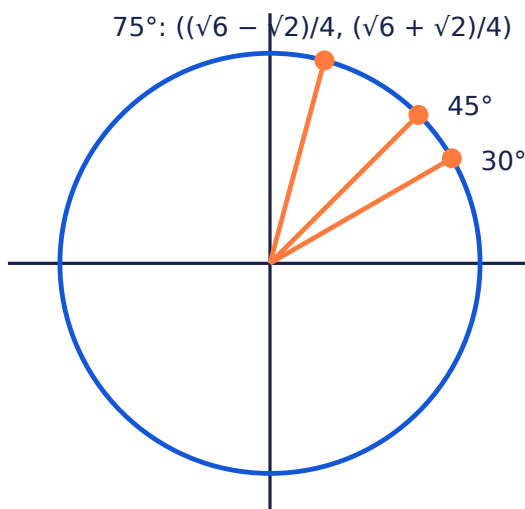
What are the sum, difference and double-angle formulas?

Angle formulas

- $\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$
- $\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$
- $\tan(A \pm B) = (\tan A \pm \tan B) \div (1 \mp \tan A \tan B)$
- $\sin 2\theta = 2 \sin \theta \cos \theta$
- $\cos 2\theta = \cos^2 \theta - \sin^2 \theta = 2\cos^2 \theta - 1 = 1 - 2\sin^2 \theta$

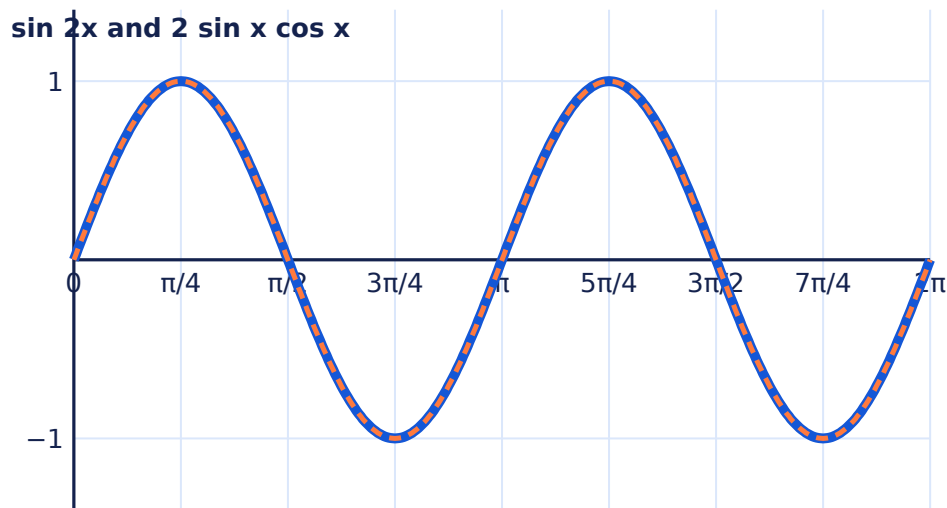
Worked example: exact value

$\sin 75^\circ = \sin(45^\circ + 30^\circ) = (\sqrt{2}/2)(\sqrt{3}/2) + (\sqrt{2}/2)(1/2) = (\sqrt{6} + \sqrt{2})/4 \approx 0.966$. The sum formula turns two special angles into a new exact value.



$$\sin 75^\circ = \sin(45^\circ + 30^\circ) = (\sqrt{6} + \sqrt{2})/4.$$

Double angles follow from the sum formulas with $A = B$. If $\sin \theta = 3/5$ with θ in Quadrant I, then $\cos \theta = 4/5$, so $\sin 2\theta = 2 \cdot (3/5)(4/5) = 24/25$ and $\cos 2\theta = 16/25 - 9/25 = 7/25$.



Both expressions draw the same curve, which illustrates the double-angle identity.

Common mistake

Assuming $\sin(A + B) = \sin A + \sin B$. With $A = B = 30^\circ$, the left side is $\sin 60^\circ \approx 0.866$ but the right side is 1.

Mathbot's tip

Test a suspicious identity with a special angle before spending time on a proof. One counterexample shows it is false.

Key points to remember

- $\sin^2 \theta + \cos^2 \theta = 1$, $1 + \tan^2 \theta = \sec^2 \theta$, $1 + \cot^2 \theta = \csc^2 \theta$.
- To verify an identity, transform one side only, usually into sines and cosines.
- $\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$; $\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$.
- $\sin 2\theta = 2 \sin \theta \cos \theta$; $\cos 2\theta = \cos^2 \theta - \sin^2 \theta = 2\cos^2 \theta - 1 = 1 - 2\sin^2 \theta$.
- $\sin(A + B)$ is not $\sin A + \sin B$; test with a special angle to catch false identities.

Proving and Using Trigonometric Identities: frequently asked questions

What is a trigonometric identity?

It is an equation involving trigonometric functions that is true for every angle where both sides are defined. For example $\sin^2 \theta + \cos^2 \theta = 1$ holds for every real θ , unlike an equation that has only certain solutions.

How do you verify a trigonometric identity?

Work on one side only. Rewrite in sines and cosines, combine fractions, expand or factor, and apply a Pythagorean identity until it matches the other side. Never move terms across the equal sign.

Where do the Pythagorean identities come from?

The point $(\cos \theta, \sin \theta)$ is on the unit circle $x^2 + y^2 = 1$, so $\cos^2 \theta + \sin^2 \theta = 1$. Dividing that identity by $\cos^2 \theta$ or $\sin^2 \theta$ gives the tangent and cotangent versions.

Why is $\sin(A + B)$ not equal to $\sin A + \sin B$?

Sine is not a linear function. A counterexample is $A = B = 30^\circ$: $\sin 60^\circ \approx 0.866$, while $\sin 30^\circ + \sin 30^\circ = 1$. The correct formula is $\sin A \cos B + \cos A \sin B$.

Which form of $\cos 2\theta$ should you use?

Use $\cos^2 \theta - \sin^2 \theta$ when both values are known, $2\cos^2 \theta - 1$ when only cosine is known, and $1 - 2\sin^2 \theta$ when only sine is known. All three are equal.