

Radians, Arc Length and the Unit Circle

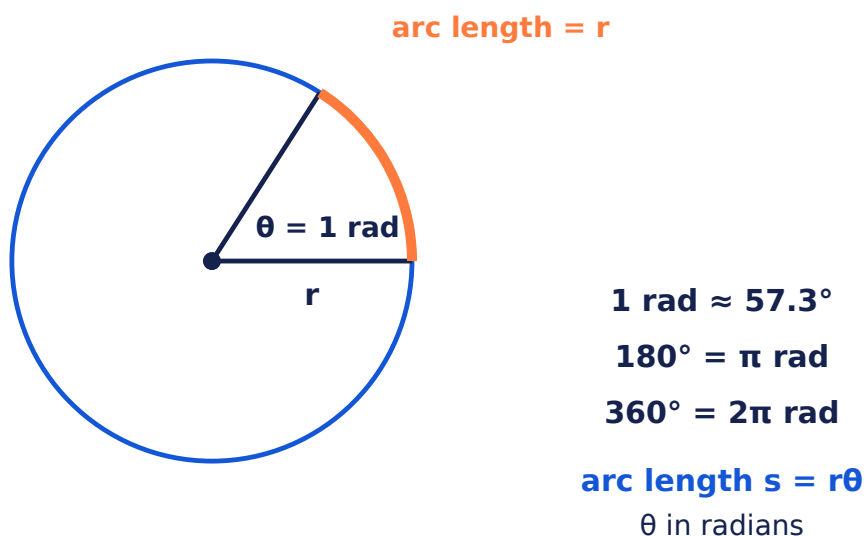
Grade 12 – Precalculus math · Lesson · Standards: F-TF.A.1, F-TF.A.2, F-TF.A.3, F-TF.A.4

Quick answer: A radian measures an angle by the arc it cuts on a circle of radius 1, so 180 degrees equals π radians. On the unit circle, the point at angle θ is $(\cos \theta, \sin \theta)$, and reference angles with symmetry give exact sine, cosine and tangent values at multiples of $\pi/6$, $\pi/4$ and $\pi/3$.

Standards: F-TF.A.1, F-TF.A.2, F-TF.A.3, F-TF.A.4 · **Level:** Grade 12 Precalculus

What is a radian, and how do you convert to degrees?

A radian is the angle at the center of a circle that cuts off an arc exactly as long as the radius. A full turn is 2π radians, so $180^\circ = \pi$ radians. Multiply degrees by $\pi/180$ to get radians, and radians by $180/\pi$ to get degrees.



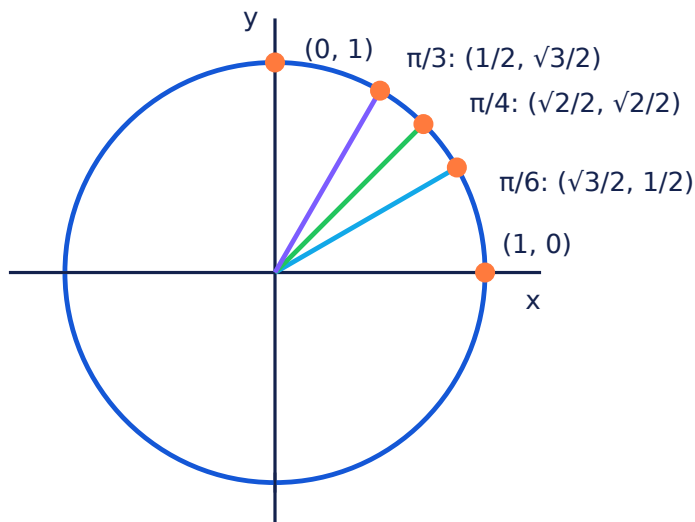
One radian is the angle whose arc is as long as the radius.

Arc length and speed

- Arc length: $s = r\theta$ (θ in radians).
- Angular speed: $\omega = \theta/t$, in radians per unit of time.
- Linear speed of a point on the rim: $v = r\omega$.

How do you read the unit circle?

On the unit circle (radius 1, center at the origin), the point reached after turning through θ from the positive x-axis is $(\cos \theta, \sin \theta)$. Then $\tan \theta = \sin \theta \div \cos \theta$. Memorize the first-quadrant values and obtain every other special angle by symmetry.



Special angles in Quadrant I: the point at angle θ is $(\cos \theta, \sin \theta)$.

Angle	cos	sin	tan
$\pi/6$	$\sqrt{3}/2$	$1/2$	$\sqrt{3}/3$
$\pi/4$	$\sqrt{2}/2$	$\sqrt{2}/2$	1
$\pi/3$	$1/2$	$\sqrt{3}/2$	$\sqrt{3}$

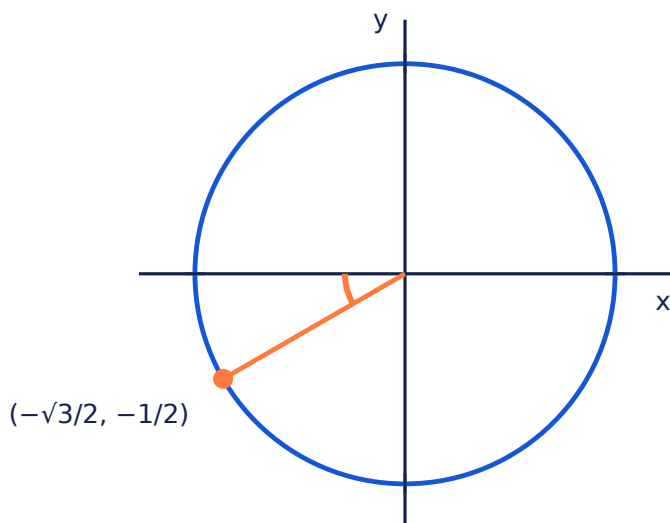
How do reference angles and symmetry give exact values?

Method: exact value at any special angle

1. Subtract multiples of 2π until the angle lies in $[0, 2\pi)$.
2. Find the quadrant and the reference angle α , the acute angle to the x-axis.
3. Use the value at α and attach the sign for the quadrant: cosine is positive in Quadrants I and IV, sine in I and II.

Worked example

Find $\cos(7\pi/6)$ and $\sin(7\pi/6)$. The angle is in Quadrant III, and $7\pi/6 - \pi = \pi/6$ is the reference angle. Both coordinates are negative, so $\cos(7\pi/6) = -\sqrt{3}/2$ and $\sin(7\pi/6) = -1/2$.



$7\pi/6$ lies in Quadrant III with reference angle $\pi/6$, so both coordinates are negative.

Sine and cosine repeat every 2π , and tangent every π . Sine is odd, $\sin(-\theta) = -\sin \theta$, while cosine is even, $\cos(-\theta) = \cos \theta$.

Common mistake

Forgetting the sign. The reference angle only gives the size of the value; the quadrant decides whether it is positive or negative.

Mathbot's tip

Picture the point on the circle first. If it is left of the y-axis, cosine is negative; if it is below the x-axis, sine is negative.

Key points to remember

- Degrees to radians: multiply by $\pi/180$; radians to degrees: multiply by $180/\pi$.
- Arc length $s = r\theta$ and linear speed $v = r\omega$, with θ in radians.
- The point at angle θ on the unit circle is $(\cos \theta, \sin \theta)$.
- Sine is odd, cosine is even; sine and cosine have period 2π , tangent has period π .
- Reference angle gives the size of a value; the quadrant gives its sign.

Radians, Arc Length and the Unit Circle: frequently asked questions

What is a radian?

It is the angle at the center of a circle that intercepts an arc equal in length to the radius. One radian is about 57.3 degrees, and a full circle is 2π radians.

How do you convert degrees to radians?

Multiply by $\pi/180$. For example, $150^\circ \times \pi/180 = 5\pi/6$. To go back, multiply radians by $180/\pi$, so $5\pi/6 \times 180/\pi = 150^\circ$.

How do you find exact trig values at special angles?

Reduce the angle to $[0, 2\pi)$, find its reference angle, recall the value at $\pi/6$, $\pi/4$ or $\pi/3$, and then choose the sign from the quadrant where the terminal side lies.

What is a reference angle?

It is the acute angle between the terminal side of an angle and the nearest part of the x-axis. For $4\pi/3$, the terminal side is $\pi/3$ past the negative x-axis, so the reference angle is $\pi/3$.

Why is cosine even and sine odd?

Turning by $-\theta$ reflects the point across the x-axis. The x-coordinate stays the same, so $\cos(-\theta) = \cos \theta$, while the y-coordinate changes sign, so $\sin(-\theta) = -\sin \theta$.